69. *Local Deformation of Pencil of Curves of Genus Two*

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§1. **Introduction.** Let S be a compact complex surface which admits a surjective holomorphic map $g: S \to \Delta$ onto a compact Riemann surface Δ. We suppose that the general fibres are smooth curves of genus 2. Then S is birationally equivalent to a branched double covering S' over a $\boldsymbol{P}^1$-bundle W over Δ whose branch locus B intersects a general $\boldsymbol{P}^1$ at 6 points. Though there are infinitely many choices of W, we can choose one, by applying elementary transformations to W, such that the branch locus B is, in some sense, canonical. After this is done, the singular fibres of g are classified into six types (0), (I_k), (II_k), (III_k), (IV_k) and (V) (see [4]). Recall that the singular fibres of type (0) are obtained by resolving only rational double points on the singular model S', and that the most general singular fibres of type (I_1) are composed of two elliptic curves with self-intersection number -1 which intersect transversally at one point (they will be called general (I_1) type).

In this paper we study deformations of surfaces with such fibration, but only locally at each singular fibre. More precisely, let $g^{-1}(P)$, $P \in \Delta$ be a singular fibre of S and let U be a small neighborhood of P and $X = g^{-1}(U)$. Then we shall prove the following theorem.

Theorem. *Assume $g^{-1}(P)$ is a singular fibre of type* (T) *other than type* (0). *Then there exists a family $\{X_t\}_{t \in M}$ of deformations of $X = X_0$, $0 \in M$ such that*

i) *each X_t admits a holomorphic map $g_t: X_t \to U$ whose general fibre is of genus* 2, *and g_t depends holomorphically on t,*

ii) *for general $t \in M$, $g_t: X_t \to U$ has only singular fibres of general* (I_1) *type and type* (0),

iii) *the number $\delta(\mathrm{T})$ of these singular fibres of general* (I_1) *type in g_t is given by*

$$\delta(\mathrm{I}_k) = \delta(\mathrm{III}_k) = 2k-1, \quad \delta(\mathrm{II}_k) = \delta(\mathrm{IV}_k) = 2k, \quad \delta(\mathrm{V}) = 1.$$

This theorem states that each singular fibre of type (T) is, in some sense, "equivalent" to $\delta(\mathrm{T})$ singular fibres of general (I_1) type modulo those of type (0). Recall that the value $\delta(\mathrm{T})$ equals the contribution of the singular fibre of type (T) to the difference $c_1^2 - (2\chi + 6(\pi - 1))$, where $\chi = \chi(\mathcal{O}_S)$, π is the genus of Δ and the Chern number c_1^2 is the value for relatively minimal S [4, Theorem 3].

The result is related to the construction of a family of deformations of elliptic double points which admits simultaneous resolution. To conclude

Introduction we want to pose the question if the same holds globally for S. Namely: Can one deform $g: S\to\Delta$ to $g_t: S_t\to\Delta_t$ whose singular fibres are all type (0) or general (I_1) type?

§ 2. Fibres of type I and II. We refer to [4] for the basic terminology about infinitely near triple points and the construction of the singular fibres as double coverings over the $\boldsymbol{P}^1$-bundles. In particular, B denotes the corresponding branch locus on a $\boldsymbol{P}^1$-bundle W and $B_0=B-$(fibres). To construct a deformation that we want, it is more convenient to pass to a slightly different model. First suppose B has singularities of type (I_k). In this case, we apply elementary transformation successively $(2k-1)$ times at one of the triple points of B_0. Then B is transformed into a divisor with $(4k-2)$ or $(4k-1)$-fold triple point Q, not containing the fibre Γ_0 through Q, and the other singularities are, if any, at most simple triple points. If B is of type (III_k), we can similarly transform it to the one with $4k$ or $(4k+1)$-fold triple point Q. We set $l=2k-1$ or $2k$.

In the both cases, B has contact of order 3 with Γ_0 at Q, and hence the second infinitely near triple point Q_1 is not on the proper transform of Γ_0. For an appropriately chosen inhomogeneous coordinate y on Γ_0, we may assume that all the infinitely near triple points are on the proper transform of $y=0$. Then the local equation for B at Q is

$$y^3+b(x)x^{4l}y+c(x)x^{6l}=0,$$

where $b(x)$, $c(x)$ are holomorphic with

$$\text{(1)} \qquad \operatorname{ord} b(x)<4 \quad \text{or} \quad \operatorname{ord} c(x)<6.$$

As a parameter space we choose a neighborhood of the origin in $\boldsymbol{C}^l$ with coordinate $(\alpha_1, \alpha_2, \cdots, \alpha_l)$ and set $h(x)=\prod_{i=1}^{l}(x-\alpha_i)$. Then we define a family $\{B_\alpha\}$, in a neighborhood of Q, by the equation

$$\text{(2)} \qquad y^3+b(x)h(x)^4y+c(x)h(x)^6=0.$$

Lemma. *Let $\{X'_\alpha\}$ be the family of surfaces in (x, y, w)-space defined by*

$$\text{(3)} \qquad w^2=y^3+b(x)h(x)^4y+c(x)h(x)^6.$$

If $4b(0)^3+27c(0)^2\neq 0$, then we can simultaneously resolve the singularities of $\{X'_\alpha\}$ (without base change).

Proof. We first blow up the ideal generated by w, y and $h(x)^2$. Let (z_0, z_1, z_2) be the homogeneous coordinates on $\boldsymbol{P}^2$ and consider the graph of $(x, y, w)\to(z_0, z_1, z_2)=(w, y, h(x)^2)$. We only need to consider two affine pieces $V_1=\{z_1\neq 0\}$ and $V_2=\{z_2\neq 0\}$. If we set $\xi_0=z_0/z_1$, $\xi_2=z_2/z_1$ on V_1, then

$$w=\xi_0 y, \quad h(x)^2=\xi_2 y, \quad \xi_0^2=y(1+b\xi_2^2+c\xi_2^3).$$

These equations define a double curve along $h(x)=\xi_0=0$.

On V_2, we set $\eta_0=z_0/z_2$, $\eta_1=z_1/z_2$. Then

$$w=\eta_0 h(x)^2, \quad y=\eta_1 h(x)^2, \quad \eta_0^2=h(x)^2(\eta_1^3+b\eta_1+c).$$

On the intersection $V_1\cap V_2$, one has $\eta_0=\xi_0/\xi_2$, $\eta_1=1/\xi_2$. So we blow up the ideal $(h(x), \xi_0)$ on V_1 and $(h(x), \eta_0)$ on V_2. Then V_2 is desingularized (modulo rational double points). As to V_1, since $\xi_2\neq 0$ is contained in V_2, we only consider a neighborhood of $\xi_2=0$. We set $(\zeta_0, \zeta_1)=(\xi_0, h(x))$. Since we only need to consider the affine piece $\zeta_0 \neq 0$, we set $u_1=\zeta_1/\zeta_0$. Then

$$w=\xi_0 y, \quad h(x)=u_1\xi_0, \quad u_1^2\xi_0^2=\xi_2 y, \quad \xi_0^2=y(1+b\xi_2^2+c\xi_2^3).$$

These equations reduce to $u_1\xi_0=h(x)$ in (x, u_1, ξ_0)-space. This is simultaneously desingularized without base change (see [5], [1]).

Similarly, we can prove:

Corollary. *Let $\{X'_\alpha\}$ be defined by* (2) *with* $\operatorname{ord} b(x)<4$ *or* $\operatorname{ord} c(x)<6$. *Then $\{X'_\alpha\}$ can be simultaneously desingularized after an appropriate base change.*

To prove our theorem for singular fibres of type (I_k) or (II_k), we construct a family $\{B_\alpha\}$ by (2), the remaining component being unchanged, and resolve the singularities. Thus we obtain a family $\{X_\alpha\}$ of smooth surfaces. If the α_i are distinct one another, then X_α is obtained from X'_α by resolving l singular points of the form $w^2=2$-fold triple point. Therefore, for general α, X_α has l singular fibres of type (I_1) at $x=\alpha_i$.

To get the fibres of general (I_1) type, we regard the constant terms b_0, c_0 of $b(x)$ and $c(x)$ as additional parameters. Then, for general values of α, b_0 and c_0, the discriminants $4b(\alpha_i)^3+27c(\alpha_i)^2$ are all non-zero. We further deform the components which are away from Q, if necessary.

§ 3. Fibres of type III, IV and V. Let B be the branch locus for the singular fibre of type (III_k). By elementary transformation at the triple point of B_0, B is transformed to a $(4k-2)$-fold triple point without containing Γ_0. Since the singular fibre of type (IV_k) comes from a $4k$-fold triple point of B, these two cases may be handled at one time, by setting $l=2k-1$ or $2k$.

Let y be a coordinate on Γ_0 and x a coordinate on U. Since the second infinitely near triple point lies on the proper transform of Γ_0, we may assume that all the infinitely near triple points are on the proper transform of the curve $y^2-x=0$.

Now the local equation for B is of the form

$$(y^2-x)^3+ax^l(y^2-x)^2+bx^{2l}(y^2-x)+cx^{3l}=0,$$

where a, b, c are holomorphic in (x, y) and of degree $\leqq 1$ in y. We take $\beta=(\beta_1, \beta_2, \cdots, \beta_l)$ as a parameter and let

$$f(y)^2=\prod_{j=1}^{l}(y-\beta_j)^2=P(y^2)+yQ(y^2).$$

We set $h(x, y)=P(x)+yQ(x)$ and define a deformation by

$$(4) \qquad (y^2-x)^3+ah(x, y)(y^2-x)^2+bh(x, y)^2(y^2-x)+ch(x, y)^3=0$$

Since $h(x, y)-f(y)^2$ is divisible by y^2-x, we can set

$$h(x, y)-f(y)^2=(y^2-x)G, \qquad G=G(x, y, \beta).$$

For $\beta=0$ we have $G(x, y, 0)=(x^l-y^{2l})/(y^2-x)=-(x^{l-1}+\cdots+y^{2l-2})$. Now (4) is written as

$$\begin{aligned}(1+aG+bG^2+cG^3)(y^2-x)^3+(a+2bG+3cG^2)f(y)^2(y^2-x)^2\\ +(b+3cG)f(y)^4(y^2-x)+cf(y)^6=0.\end{aligned}$$

We can use $z=y^2-x$ and y as local coordinates and the above equation shows that, for general β, B has 2-fold triple points at $(z, y)=(0, \beta_i)$, that is, at $(x, y)=(\beta_i^2, \beta_i)$, $(i=1, 2, \cdots, l)$.

Let $\{X'_\beta\}$ be the family of double coverings with branch loci $\{B_\beta\}$. Then the singularities can be simultaneously desingularized to $\{X_\beta\}$. For general β, X_β has l singular fibres of type (I_1).

For a singular fibre of type (V), the branch locus is defined by the equation $x(y^6+axy^4+bx^2y^2+cx^4)=0$. If we apply elementary transformation at $(x, y)=(0, 0)$, this is transformed into

$$x(x^2+axy^2+by^4+cy^6)=0.$$

We define a family with three parameters (t, s, α) by

$$(x-ty^2)^3+a(x-ty^2)^2(y-\alpha)^2+b(x-ty^2)(y-\alpha)^4+(cx+s)(y-\alpha)^6=0.$$

For $t\neq 0$, this has a 2-fold triple point at $(x, y)=(t\alpha^2, \alpha)$, and determines a singular fibre of type (I_1). The double coverings with these branch loci can be simultaneously desingularized by canonical resolution as in [3, § 2].

This completes the proof of the theorem.

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