64. A Generalization of the Hille-Yosida Theorem

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1. Introduction. Let X be a Banach space, and let $B(X)$ be the set of all bounded linear operators from X into itself. Arendt [1] introduced the notion of integrated semigroups and obtained the following generalization of the Hille-Yosida theorem: A closed linear operator A is the generator of a once integrated semigroup $\{U(t); t\geq 0\}$ on X satisfying $\|U(t+h)-U(t)\|\leq Mhe^{a(t+h)}$ for $t, h\geq 0$ if and only if $(a, \infty)\subset\rho(A)$ and $\|(\lambda-A)^{-m}\|\leq M/(\lambda-a)^m$ for $\lambda>a$ and $m\geq 1$, where $M>0$ and $a\geq 0$ are constants. Moreover, the part of A in $\overline{D(A)}$ is the generator of a (C_0)-semigroup on $\overline{D(A)}$.

Let $C\in B(X)$ be injective. In this paper we introduce the notion of integrated C-semigroups and prove the following theorems.

Theorem 1. *An operator A is the generator of an integrated C-semigroup $\{U(t); t\geq 0\}$ on X satisfying*

(1.1) $$\|U(t+h)-U(t)\|\leq Mhe^{a(t+h)} \qquad \text{for } t, h\geq 0,$$

where $M>0$ and $a\geq 0$ are constants, if and only if A satisfies the following properties (A1)–(A3) *and it is maximal with respect to* (A1)–(A3):

(A1) *A is a closed linear operator and $\lambda-A$ is injective for $\lambda>a$;*

(A2) *$D((\lambda-A)^{-m})\supset R(C)$ and $\|(\lambda-A)^{-m}C\|\leq M/(\lambda-a)^m$ for $\lambda>a$ and $m\geq 1$;*

(A3) *$Cx\in D(A)$ and $ACx=CAx$ for $x\in D(A)$.*

Theorem 2. *If A satisfies the equivalent conditions of Theorem* 1, *then the part of A in $\overline{D(A)}$ is the generator of a C_1-semigroup $\{S_1(t); t\geq 0\}$ on $\overline{D(A)}$ satisfying $\|S_1(t)x\|\leq Me^{at}\|x\|$ for $x\in\overline{D(A)}$ and $t\geq 0$, where $C_1=C|_{\overline{D(A)}}$.*

The above-mentioned Arendt's results are the case of $C=I$ (the identity) in Theorems 1 and 2. As direct consequences of Theorems 1 and 2 we have:

Corollary 1. *If A satisfies* (A1)–(A3) *in Theorem* 1 *then $C^{-1}AC$ is the generator of an integrated C-semigroup $\{U(t); t\geq 0\}$ on X satisfying $\|U(t+h)-U(t)\|\leq Mhe^{a(t+h)}$ for $t, h\geq 0$.*

Corollary 2 ([2, Corollary 13.2]). *Suppose $\overline{R(C)}=X$. A is the generator of a C-semigroup $\{S(t); t\geq 0\}$ on X satisfying $\|S(t)\|\leq Me^{at}$ for $t\geq 0$ if and only if A is maximal with respect to* (A2), (A3) *in Theorem* 1 *and* "(A1′) *A is a closed linear operator with $\overline{D(A)}=X$ and λ-A is injective for $\lambda>a$*".

2. Integrated C-semigroups. Let $C\in B(X)$ be injective. A family $\{U(t); t\geq 0\}$ in $B(X)$ is called an *integrated C-semigroup on X*, if

(2.1) $U(\cdot)x: [0, \infty)\to X$ is continuous for $x\in X$,

(2.2) $U(t)x=0$ for all $t>0$ implies $x=0$,

(2.3) there exist $K>0$ and $b\geq 0$ such that $\|U(t)\|\leq Ke^{bt}$ for $t\geq 0$,

(2.4) $U(0)=0$ (the zero operator) and $U(t)C=CU(t)$ for $t>0$, (2.5) $U(t)U(s)x=\int_t^{s+t} U(r)Cx\,dr-\int_0^s U(r)Cx\,dr$ for $x \in X$ and $t, s \geq 0$.

Let $\{U(t);\, t \geq 0\}$ be an integrated C-semigroup on X. For $\lambda > \omega_0 \equiv \max\{0, \overline{\lim}_{t\to\infty} (\log \| U(t)\|)/t\}$ we define $L(\lambda) \in B(X)$ by

$$L(\lambda)x=\int_0^\infty \lambda e^{-\lambda t} U(t)x\,dt \qquad \text{for } x \in X.$$

Clearly $L(\lambda)C=CL(\lambda)$ for $\lambda > \omega_0$, and a simple computation yields

$$L(\mu)C-L(\lambda)C=(\lambda-\mu)L(\lambda)L(\mu) \qquad \text{for } \lambda, \mu > \omega_0.$$

It follows from this that $L(\mu)$ is injective for $\mu > \omega_0$ and the following holds:

(2.6)
$$\{x \in X;\, Cx \in R(L(\lambda))\}=\{x \in X;\, Cx \in R(L(\mu))\} \quad (\equiv D(A)),$$
$$(\lambda-L(\lambda)^{-1}C)x=(\mu-L(\mu)^{-1}C)x \qquad \text{for } \lambda, \mu > \omega_0 \text{ and } x \in D(A).$$

Therefore the closed linear operator A defined by

$$Ax=(\lambda-L(\lambda)^{-1}C)x \qquad \text{for } x \in D(A) \equiv \{x \in X;\, Cx \in R(L(\lambda))\}$$

is independent of $\lambda > \omega_0$. The operator A is called the *generator* of the integrated C-semigroup $\{U(t);\, t \geq 0\}$. The generator has the following

(2.7) $Cx \in D(A)$ and $ACx=CAx$ for $x \in D(A)$,

(2.8)
$$(\lambda-A)L(\lambda)x=Cx \qquad \text{for } x \in X \text{ and } \lambda > \omega_0$$
$$L(\lambda)(\lambda-A)x=Cx \qquad \text{for } x \in D(A) \text{ and } \lambda > \omega_0.$$

Example. Let Z be the generator of a C-semigroup $\{S(t);\, t \geq 0\}$ on X with $\|S(t)\| \leq Me^{at}$ for $t \geq 0$, where $M>0$ and $a \geq 0$ are constants. (We refer to [2, 4] for C-semigroups.) Define $U(t) \in B(X)$ for $t \geq 0$ by

$$U(t)x=\int_0^t S(s)x\,ds \qquad \text{for } x \in X.$$

Then $\{U(t);\, t \geq 0\}$ is an integrated C-semigroup on X whose generator is Z, and $\| U(t+h)-U(t)\| \leq Mhe^{a(t+h)}$ for $t, h \geq 0$.

Lemma. *Let A be the generator of an integrated C-semigroup $\{U(t);\, t \geq 0\}$ on X. Then for $t \geq 0$ we have*

(2.9) $AU(t)x=U(t)Ax$ *and* $U(t)x=tCx+\int_0^t U(s)Ax\,ds$ *for* $x \in D(A)$,

(2.10) $\int_0^t U(s)x\,ds \in D(A)$ *and* $U(t)x=tCx+A\int_0^t U(s)x\,ds$ *for* $x \in X$.

Moreover, if $\{U(t);\, t \geq 0\}$ *satisfies* (1.1) *then* A *satisfies* (A1)–(A3).

Proof. By (2.8), $\lambda-A$ is injective and

$$(\lambda-A)^{-1}Cx=\lambda\int_0^\infty e^{-\lambda t}U(t)x\,dt \qquad \text{for } \lambda > \omega_0 \text{ and } x \in X.$$

Set $f(\lambda, x)=\lambda^{-1}(\lambda-A)^{-1}Cx$ for $x \in X$ and $\lambda > \omega_0$. The Post-Widder inversion formula [3, Theorem 6.3.5] implies

(2.11) $U(t)x=\lim_{m\to\infty}((-1)^m/m!)(m/t)^{m+1}f^{(m)}(m/t, x)$ for $x \in X$ and $t>0$.

Let $x \in D(A)$. Then $f(\lambda, Ax)=Af(\lambda, x)$ by (2.7), which implies $f^{(m)}(\lambda, Ax)=Af^{(m)}(\lambda, x)$ for $\lambda > \omega_0$ and $m \geq 0$. Combining this with (2.11) we see that $U(t)x \in D(A)$ and $AU(t)x=U(t)Ax$ for $t \geq 0$. Similarly as in the proof of [1, Proposition 3.3], we obtain the latter half of (2.9) and (2.10).

Suppose that $\{U(t);\, t \geq 0\}$ satisfies (1.1). Since $a \geq \omega_0$ by $\| U(t)\| \leq Mte^{at}$, A satisfies (A1), (A3) (=(2.7)) and

(2.12) $$(\lambda-A)^{-1}Cx=\int_0^\infty \lambda e^{-\lambda t}U(t)x\,dt \qquad \text{for } x \in X \text{ and } \lambda > a.$$

Moreover, by induction on m we obtain for $x \in X$, $\lambda > a$ and $m \geq 2$

$$(\lambda - A)^{-m}Cx = (C^{-1}L(\lambda))^m Cx$$

$$(2.13)\qquad = \int_0^\infty \cdots \int_0^\infty \lambda e^{-\lambda(t_1+\cdots+t_m)}(U(t_1+\cdots+t_m)x - U(t_2+\cdots+t_m)x)dt_m \cdots dt_1.$$

Let $x \in X$, $x^* \in X^*$ (the dual of X), $\lambda > a$ and $m \geq 1$. By (1.1), $x^*(U(t)x)$ is differentiable and $|(d/dt)x^*(U(t)x)| \leq Me^{at}\|x^*\|\|x\|$ for a.e. t. Therefore

$$|x^*(U(t+h)x - U(t)x)| = \left|\int_t^{t+h} (d/ds)x^*(U(s)x)\,ds\right| \leq M\|x^*\|\|x\|\int_0^h e^{a(t+s)}ds \qquad \text{for } t, h \geq 0.$$

Combining this with (2.13) and (2.12) we obtain (A2). Q.E.D.

3. Proof of Theorems.

Proof of Theorem 1. (Necessity) Let A be the generator of an integrated C-semigroup $\{U(t); t \geq 0\}$ on X satisfying (1.1). A satisfies (A1)–(A3) by Lemma. Suppose that $\mathfrak{A}$ satisfies (A1)–(A3) with A replaced by $\mathfrak{A}$ and $\mathfrak{A} \supset A$. To show $\mathfrak{A} = A$, let $x \in D(\mathfrak{A})$ and set $f(\lambda, z) = \lambda^{-1}(\lambda - A)^{-1}Cz$ for $z \in X$ and $\lambda > \omega_0$. For $\lambda > \omega_0$, $f(\lambda, \mathfrak{A}x) = \lambda^{-1}(\lambda - \mathfrak{A})^{-1}C\mathfrak{A}x = \mathfrak{A}\lambda^{-1}(\lambda - \mathfrak{A})^{-1}Cx = \mathfrak{A}f(\lambda, x)$ which implies $\mathfrak{A}f^{(m)}(\lambda, x) = f^{(m)}(\lambda, \mathfrak{A}x)$ for $m \geq 0$. Combining this with (2.11) we see that $U(t)x \in D(\mathfrak{A})$ and $\mathfrak{A}U(t)x = U(t)\mathfrak{A}x$ for $t \geq 0$. Hence

$$Cx = (\lambda - \mathfrak{A})\int_0^\infty \lambda e^{-\lambda t}U(t)x\,dt = \int_0^\infty \lambda e^{-\lambda t}U(t)(\lambda - \mathfrak{A})x\,dt = L(\lambda)(\lambda - \mathfrak{A})x$$

for $\lambda > \omega_0$, which implies $x \in D(A)$. Thus $\mathfrak{A} = A$.

(Sufficiency) For $x \in X$ and $m \geq 1$,

$$R(\lambda)^m Cx = \sum_{l=m-1}^{\infty} {}_lC_{m-1}(\mu - \lambda)^{l-m+1}R(\mu)^{l+1}Cx \qquad \text{for } \mu > \lambda > a,$$

where $R(\lambda) = (\lambda - A)^{-1}$, which implies $(d/d\lambda)R(\lambda)^m Cx = -mR(\lambda)^{m+1}Cx$ for $\lambda > a$. Now, by induction on m we obtain that for $x \in X$, $\lambda > a$ and $m \geq 1$,

$$(3.1)\qquad (d/d\lambda)^m(\lambda - A)^{-1}Cx = m!\,(-1)^m(\lambda - A)^{-(m+1)}Cx.$$

Hence by (A2), $\|(d/d\lambda)^m(\lambda - A)^{-1}Cx\| \leq m!\,M\|x\|/(\lambda - a)^{m+1}$ for $x \in X$, $\lambda > a$ and $m \geq 0$. By [1, Corollary 1.2] there exists a family $\{U(t); t \geq 0\}$ in $B(X)$ such that $U(0) = 0$, $\|U(t+h) - U(t)\| \leq Mhe^{a(t+h)}$ for $t, h \geq 0$ and

$$(3.2)\qquad (\lambda - A)^{-1}Cx = \int_0^\infty \lambda e^{-\lambda t}U(t)x\,dt \qquad \text{for } x \in X \text{ and } \lambda > a.$$

Clearly $\{U(t); t \geq 0\}$ satisfies (2.1)–(2.3). Since (A3) is equivalent to

(A3′) $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$ for $\lambda > a$ and $x \in D((\lambda - A)^{-1})$,

it follows from (3.2) that

$$\int_0^\infty \lambda e^{-\lambda t}U(t)Cx\,dt = \int_0^\infty \lambda e^{-\lambda t}CU(t)x\,dt \qquad \text{for } x \in X \text{ and } \lambda > a.$$

By the uniqueness theorem for Laplace transforms we see that $U(t)C = CU(t)$ for $t \geq 0$, i.e., (2.4) holds. Since $(\lambda - A)^{-1}Cx - (\mu - A)^{-1}Cx = (\mu - \lambda)(\lambda - A)^{-1}(\mu - A)^{-1}Cx$ for $x \in X$ and $\lambda, \mu > a$, similarly as in the proof of [1, Theorem 3.1] we see that (2.5) holds.

Let $\tilde{A}$ be the generator of $\{U(t); t \geq 0\}$. To see $A \subset \tilde{A}$, let $x \in D(A)$ and put $(\lambda - A)x = y$, where $\lambda > a \geq \omega_0$. By (A3′) and (3.2) we see that

$$Cx = (\lambda - A)^{-1}Cy = \int_0^\infty \lambda e^{-\lambda t}U(t)y\,dt = L(\lambda)y.$$

Hence $x \in D(\tilde{A})$ and $\tilde{A}x = \lambda x - L(\lambda)^{-1}Cx = Ax$. This means $A \subset \tilde{A}$. Hence we obtain $A = \tilde{A}$, because $\tilde{A}$ satisfies (A1)–(A3) with A replaced by $\tilde{A}$ from Lemma and A is maximal with respect to the properties (A1)–(A3). Q.E.D.

Proof of Theorem 2. As in the proof of [1, Corollary 4.2] we see that $U(\cdot)x \in C^1([0,\infty), X)$ and $(d/dt)U(t)x \in \overline{D(A)}$ for $x \in \overline{D(A)}$ and $t \geq 0$.

Now, for $t \geq 0$, define $S_1(t): \overline{D(A)} \to \overline{D(A)}$ by $S_1(t)x = (d/dt)U(t)x$ for $x \in \overline{D(A)}$. Then $\{S_1(t); t \geq 0\}$ is a C_1-semigroup on $\overline{D(A)}$ satisfying (3.3)–(3.6), where $C_1 = C|_{\overline{D(A)}}$;

(3.3) $\|S_1(t)x\| \leq Me^{at}\|x\|$ for $x \in \overline{D(A)}$,

(3.4) $\|S_1(t+h)x - S_1(t)x\| \leq Mhe^{a(t+h)}\|Ax\|$ for $x \in D(A)$ and $t, h \geq 0$,

(3.5) $S_1(t)x = C_1x + A_1\int_0^t S_1(s)x\,ds$ for $x \in \overline{D(A)}$ and $t \geq 0$,

(3.6) $(\lambda - A_1)\int_0^\infty e^{-\lambda t}S_1(t)x\,dt = C_1x$ for $x \in \overline{D(A)}$ and $\lambda > a$.

In fact, (3.3)–(3.5) follow from (1.1), (2.9) and (2.10). Differentiating

$$U(t)U(s)x = \int_t^{s+t} U(r)C_1x\,dr - \int_0^s U(r)C_1x\,dr$$

with respect to s and t, we get $S_1(t)S_1(s)x = S_1(s+t)C_1x$ for $\overline{D(A)}$ and $t, s \geq 0$. Thus $\{S_1(t); t \geq 0\}$ is a C_1-semigroup on $\overline{D(A)}$. (3.6) follows from (2.8).

Finally, let A_1 be the part of A in $\overline{D(A)}$ and let Z_1 be the generator of the C_1-semigroup $\{S_1(t); t \geq 0\}$ on $\overline{D(A)}$. To see $A_1 = Z_1$, let $x \in D(A_1)$. $A_1x = Ax \in \overline{D(A)}$ and $AU(t)x = U(t)A_1x$ imply $S_1(t)x \in D(A_1)$ and $A_1S_1(t)x = S_1(t)A_1x$. From this and (3.6) we see that for $\lambda > a$

$$C_1x = \int_0^\infty e^{-\lambda t}S_1(t)(\lambda - A_1)x\,dt = \mathfrak{L}_\lambda(\lambda - A_1)x, \quad \text{where } \mathfrak{L}_\lambda z = \int_0^\infty e^{-\lambda t}S_1(t)z\,dt$$

for $z \in \overline{D(A)}$ and $\lambda > a$. From the definition of generator it follows that $Z_1x \equiv (\lambda - \mathfrak{L}_\lambda^{-1}C_1)x = A_1x$. Thus we get $A_1 \subset Z_1$. Next, as in the proof of [4, Theorem 2.1] we have $Z_1 \subset C_1^{-1}A_1C_1$. Moreover $C_1^{-1}A_1C_1 \subset C^{-1}AC = A$, because $C^{-1}AC$ satisfies (A1)–(A3) with A replaced by $C^{-1}AC$ and $A \subset C^{-1}AC$ by (A3). This implies $C_1^{-1}A_1C_1 \subset A_1$. Hence $Z_1 \subset A_1$. Q.E.D.

Proof of Corollary 1. By the maximal principle there is an $\tilde{A} \supset A$ such that $\tilde{A}$ is maximal with respect to (A1)–(A3). We see that $\tilde{A} \subset C^{-1}AC$ and $C^{-1}AC$ satisfies (A1)–(A3). Therefore $C^{-1}AC = \tilde{A}$. Q.E.D.

Corollary 2 follows from [2, Proposition 7], Example and Theorems 1, 2.

Remark. Let A satisfy (A1)–(A3). A is maximal with respect to the properties (A1)–(A3) if and only if $A = C^{-1}AC$.

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