7. Parabolic Components of Zeta Functions

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(Communicated by Kunihiko KODAIRA, M. J. A., Jan. 12, 1988)

The functional equation for the Riemann zeta function $\zeta(s)$ was discovered by Euler [1] in 1749 in the form $\zeta(1-s)=\Gamma_{C}(s)\cos(\pi s/2)\zeta(s)$ with $\Gamma_{C}(s)=2(2\pi)^{-s}\Gamma(s)$. Later, Riemann [2] found the symmetric functional equation: $\Gamma_{R}(s)\zeta(s)=\Gamma_{R}(1-s)\zeta(1-s)$ where $\Gamma_{R}(s)=\pi^{-s/2}\Gamma(s/2)$. These two functional equations are equivalent since $\Gamma_{R}(s)\Gamma_{R}(s+1)=\Gamma_{C}(s)$ and $\Gamma_{R}(1+s)\Gamma_{R}(1-s)=(\cos(\pi s/2))^{-1}$, but as is well-known the Riemann's form has been more suggestive to later developments of arithmetic zeta functions containing the adelic view point, where $\Gamma_{R}(s)\zeta(s)$ is considered as the product of local zeta functions.

The same is true for Selberg zeta functions. Let M be a compact Riemann surface of genus $g\geqq 2$, and $\Gamma=\pi_1(M)$ the fundamental group embedded in $\mathrm{PSL}_2(\boldsymbol{R})$, so $M=\Gamma\backslash H$ for the upper half plane H. The zeta function $Z_{\mathrm{hyp}}(s)$ of Γ (or M) is defined by Selberg [3] as the product over all primitive hyperbolic conjugacy classes of Γ. The functional equation of Selberg was not symmetric corresponding to Euler. Later, Vignéras [4] as Riemann presented the symmetric functional equation $Z_{\mathrm{id}}(s)Z_{\mathrm{hyp}}(s)=Z_{\mathrm{id}}(1-s)Z_{\mathrm{hyp}}(1-s)$ with the identity factor $Z_{\mathrm{id}}(s)=\Gamma_2^{C}(s)^{2g-2}=((2\pi)^s\Gamma_2(s)^2\Gamma(s)^{-1})^{2g-2}$ where $\Gamma_2(s)$ is the double gamma function of Barnes. Recently, Voros [5] and Sarnak [6] give the determinant expression

$$Z_{\mathrm{id}}(s)Z_{\mathrm{hyp}}(s)=\det(\Delta-s(1-s))\exp((2g-2)(C+s(1-s)))$$

where Δ is the Laplace operator acting on $L^2(M)$ and $C=-1/4-(1/2)\log(2\pi)+2\zeta'(-1)$. Letting $s\to 1$, they reprove

$$Z'_{\mathrm{hyp}}(1)=\det{}'(\Delta)\exp((2g-2)(C+\log(2\pi)))$$

which was previously shown by string physicists.

We study the case of non-co-compact Γ (non-compact M). The basic case is $\Gamma=\mathrm{PSL}_2(\boldsymbol{Z})$, and hereafter we treat this case since the general feature appears explicitly here. The case of congruence subgroups is quite similar and our method is directly applicable. According to Vignéras [4] we have the symmetric functional equation

$$Z_{\mathrm{hyp}}(s)Z_{\mathrm{id}}(s)Z_{\mathrm{ell}}(s)Z_{\mathrm{par}}(s)=Z_{\mathrm{hyp}}(1-s)Z_{\mathrm{id}}(1-s)Z_{\mathrm{ell}}(1-s)Z_{\mathrm{par}}(1-s)$$

with $Z_{\mathrm{id}}(s)=\Gamma_2^{C}(s)^{1/6}$. Unfortunately $Z_{\mathrm{ell}}(s)$ and $Z_{\mathrm{par}}(s)$ are incompletely (or erroneously) defined in [4]. In the remarkable paper [7], Fischer gives correctly

$$Z_{\mathrm{ell}}(s)=\Gamma(s/2)^{-1/2}\Gamma((s+1)/2)^{1/2}\Gamma(s/3)^{-2/3}\Gamma((s+2)/3)^{2/3}$$

and $Z_{\mathrm{par}}(s)$ a bit implicitly; we refer to Venkov [8] for related calculations. More precisely we have

Theorem 1. $Z_{\mathrm{par}}(s)=(\Gamma_{\boldsymbol{R}}(2s)\zeta(2s)\Gamma(s+1/2)2^{s})^{-1}$.

Proof. It is sufficient to correct [4, p. 245]: we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\zeta'}{\zeta}(-2ir)\frac{dr}{r^2+z^2}=\frac{1}{z}\frac{\zeta'}{\zeta}(-2z)+\sum_{k=1}^{\infty}\frac{1}{z^2-k^2}$$

and add $1/2z^2$ since $\varphi(1/2)=-1$. Q.E.D.

Theorem 2.

(1) $Z_{\mathrm{hyp}}(s)Z_{\mathrm{id}}(s)Z_{\mathrm{ell}}(s)Z_{\mathrm{par}}(s)=\det(\Delta-s(1-s))\exp(a+s(1-s)b)$ *for constants a and b, where Δ is the Laplace operator of* $\mathrm{PSL}_2(\boldsymbol{Z})$.

(2) $Z'_{\mathrm{hyp}}(1)=\det'(\Delta)\zeta(2)e^{a}2^{-1/6}\pi^{-5/12}\Gamma(1/3)^{2/3}$.

Proof. The finiteness of $\det(\Delta-s(1-s))$ is shown in Theorem 3 below. Let $F_1(s)$ (resp. $F_2(s)$) be the left (resp. right) hand side. Then we have

$$\frac{1}{2s-1}\frac{d}{ds}\left(\frac{1}{2s-1}\frac{d}{ds}\log F_i(s)\right)=-\sum_{n=0}^{\infty}(\lambda_n-s(1-s))^{-2}$$

for $i=1, 2$ where $0=\lambda_0<\lambda_1\leqq\lambda_2\leqq\cdots$ denote the eigenvalues of Δ. The case $i=1$ follows from the trace formula, and the case $i=2$ is simply obtained from the definition

$$\det(\Delta-s(1-s))=\exp\left(-\frac{\partial}{\partial z}\sum_{n=0}^{\infty}(\lambda_n-s(1-s))^{-z}\Big|_{z=0}\right).$$ Q.E.D.

It is remarkable that the completed zeta function of $\mathrm{PSL}_2(\boldsymbol{Z})$ is consequently (gammas) $\zeta(2s)^{-1}Z_{\mathrm{hyp}}(s)$. More generally when the scattering determinant $\varphi(s)$ (for the datum (Γ, ρ) consisting of a discrete group Γ and a finite dimensional unitary representation ρ of Γ) is given by $L(2-2s)/L(2s)$ then $Z_{\mathrm{par}}(s, (\Gamma, \rho))$ is essentially given by $L(2s)^{-1}$ and $Z_{\mathrm{par}}(s, (\Gamma, \rho))Z_{\mathrm{par}}(1-s, (\Gamma, \rho))^{-1}$ coincides with $\varphi(s)$ up to elementary factors.

Theorem 3. (1) *The spectral zeta function $\zeta(s, \Delta)=\sum_{n=1}^{\infty}\lambda_n^{-s}$ is holomorphic on* $\boldsymbol{C}$ *except for the following poles: a simple pole at $s=1$, and double poles at $s=(1/2)-k$ for $k=0, 1, 2, \cdots$.*

(2) $$\sum_{n=1}^{\infty}e^{-t\lambda_n}\sim\frac{c_{-2}}{t}+\frac{c_{-1,1}\log t+c_{-1,0}}{t^{1/2}}+c_0+\sum_{n=1}^{\infty}(c_{n,0}+c_{n,1}\log t)t^{n/2}\ \text{as}\ t\to+0,$$

where $c_{-2}=1/12$ and $c_{n,1}=0$ for even n.

Proof. We show (2), then (1) follows by the Mellin transformation. We use the trace formula of Selberg [3] in the form of Hejhal [9, p. 510] combined with the method of Delsarte [10]. (The case of a congruence subgroup is similar by using the trace formula of Hejhal [9, Chap. 11] and Huxley [11].) Taking $h(r)=\exp(-(1/4+r^2)t)$ in the trace formula we have the decomposition $\sum_{n=0}^{\infty}e^{-t\lambda_n}=H(t)+I(t)+E(t)+(P_1(t)+P_2(t)+P_3(t))$ into hyperbolic, identity, elliptic, and parabolic components, where

$$P_1(t)=\log\left(\frac{\pi}{2}\right)\frac{1}{2\sqrt{\pi t}}e^{-t/4},$$

$$P_2(t)=2\sum_{q}\frac{\Lambda(q)}{q}\frac{1}{2\sqrt{\pi t}}\exp\left(-\left(\frac{t}{4}+\frac{(\log q)^2}{t}\right)\right),$$

$$P_3(t)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(-(1/4+r^2)t)(\psi(1+ir)+\psi(1/2+ir))dr$$

where $\psi(z)=\Gamma'(z)/\Gamma(z)$ and q runs over all prime powers. It is easy to see that $H(t)$ and $P_2(t)$ are rapidly decreasing (than any positive power of t) as $t\to+0$. In $E(t)$ and $P_1(t)$ we can expand the exponential containing t and we have power series of t. For $I(t)$ we use

$$I(t)=\frac{\pi}{12t}\int_0^\infty \exp(-(1/4+r^2)t)\,\mathrm{sech}^2(\pi r)dr$$

and expand it in t. Lastly we look at the most crucial $P_3(t)$. By using Mellin transformation, it is sufficient to show that for positive constants a and b the function

$$Q(s)=\int_0^\infty\left(\frac{1}{4}+r^2\right)^{-s}\mathrm{Re}\,\psi(a+ibr)dr$$

is holomorphic on $\boldsymbol{C}$ except for double poles at $s=(1/2)-k$ for $k=0, 1, 2, \cdots$. Splitting the integral at $r=1/2$ we see

$$Q(s)=4^{s-1}\int_0^1(1+t)^{-s}t^{-1/2}\,\mathrm{Re}\,\psi(a+i(b\sqrt{t}/2))dt$$
$$+4^{s-1}\int_0^1(1+x)^{-s}x^{s-3/2}\,\mathrm{Re}\,\psi(a+i(b/2\sqrt{x}))dx,$$

where the former is holomorphic on $\boldsymbol{C}$. The Stirling-Binet formula (Whittaker-Watson [12, p. 252]) shows that

$$\mathrm{Re}\,\psi(a+ibr)=\log r+\log b+\frac{1}{12b^2r^2}+\cdots+\frac{c_N}{r^{2N}}+R_N(r)$$

with $|R_N(r)|\leqq M_N/r^{2N+1}$ in $r\geqq 1/2$, where c_n are expressed via Bernoulli numbers. Since the integral of the remainder term is holomorphic in $\mathrm{Re}(s)>-N$, it is sufficient to notice that

$$\int_0^1(1+x)^{-s}x^{s-3/2}(\log x)^m x^n\mathrm{d}x=\sum_{k=0}^\infty\binom{-s}{k}\frac{(-1)^m m!}{(s+n+k-1/2)^{m+1}}$$

for $m, n=0, 1, 2, \cdots$, and $\left|\binom{-s}{k}\right|\sim\left|\Gamma(s)^{-1}\right|k^{\mathrm{Re}(s)-1}$ as $k\to\infty$. Q.E.D.

We notice that (1) is extended to $\sum_{n=0}^\infty(\lambda_n-s(1-s))^{-z}$ by applying the Mellin transformation to (2) multiplied by $e^{ts(s-1)}$; the holomorphy at $z=0$ implies the finiteness of $\det(\Delta-s(1-s))$ needed in Theorem 2. We remark that Theorem 3 has an independent interest from the view point of spectral geometry, since we do not have much knowledge concerning the spectral zeta function and the associated asymptotic expansion in the case of non-compact Riemannian spaces such as $\mathrm{PSL}_2(\boldsymbol{Z})\backslash H$ in general. The appearance of double poles in (1) and the appearance of $\log t$ in (2) are characteristic in comparison with the compact case. Similar results are obtained in some higher rank non-compact locally symmetric spaces also since the crucial point is the study of the scattering determinant. As noted in [17], Theorems 1 and 2 indicate the symbiosis-evolutional interpretation of the parabolic components as "chloroplasts"; see [13]–[17]. We supplement [17] by noting that we have similarly

$$s(s-1)\Gamma_F(s)\zeta_F(s)=C\cdot\det(\Delta_F-s(1-s))$$

with a constant C. Here F is a finite extension field of $\boldsymbol{Q}$, $\zeta_F(s)$ is the

Dedekind zeta function, $\Gamma_F(s)=D(F)^{s/2}\Gamma_R(s)^{r_1}\Gamma_C(s)^{r_2}$ with the absolute value $D(F)$ of the discriminant, and Δ_F is the infinite diagonal matrix diag $(\lambda_1(F), \lambda_2(F), \cdots)$ with $\lambda_n(F)=1/4+r_n(F)^2$, where $0\leqq r_1(F)\leqq r_2(F)\leqq\cdots$ run over non-negative numbers satisfying $\zeta_F(1/2+ir_n(F))=0$ assuming the Riemann hypothesis for $\zeta_F(s)$. It is sufficient to apply $(2s-1)^{-1}d\log$ to the both sides by noting that the poles of the spectral zeta function $\zeta(s, \Delta_F)=\sum_{n=1}^{\infty}\lambda_n(F)^{-s}$ are double poles at $s=(1/2)-k$ for $k=0, 1, 2, \cdots$ by the above property of $Q(s)$ since the explicit formula for $\zeta_F(s)$ is exactly similar to the parabolic term treated above. We notice that the "maximal pole" is also double which is distinct from the case of the spectral zeta function of a finite dimensional Riemannian space of finite volume where the maximal pole would be (at most) simple as in Theorem 3. In infinite dimensional or infinite volume cases we might have the maximal pole of multiple order. We refer to Tamura [18] for the $\log(\infty)$ volume case; for example, let $M=\{(x, y)\in \boldsymbol{R}^2;\ 0<y<a/\sqrt{x^2+1}\}$ for a positive constant a, then the spectra of the Laplace operator $\Delta_M=-(\partial^2/\partial x^2+\partial^2/\partial y^2)$ are all discrete (point spectra) and

$$N(x, \Delta_M)=\#\{\lambda: \text{eigenvalues};\ 0<\lambda<x\}\sim\frac{a}{4\pi^2}x\log x$$

as $x\to\infty$, which suggests that $\zeta(s, \Delta_M)=\sum_\lambda \lambda^{-s}$ has the double (maximal) pole at $s=1$.

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