60. *Exponentials of Certain Completions of the Unitary Form of a Kac-Moody Algebra*

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Let $\mathfrak{g}_{\boldsymbol{R}}$ be a real Kac-Moody algebra corresponding to a symmetrizable generalized Cartan matrix (=GCM) A, and $\mathfrak{h}_{\boldsymbol{R}}$ be the Cartan subalgebra of $\mathfrak{g}_{\boldsymbol{R}}$. Then, $\mathfrak{g}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{g}_{\boldsymbol{R}}$ is the complex Kac-Moody algebra with the same GCM A, and $\mathfrak{h}=\boldsymbol{C}\otimes_{\boldsymbol{R}}\mathfrak{h}_{\boldsymbol{R}}$ is the Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{k}$ be the unitary form of $\mathfrak{g}$ given in [2]. In [3], we considered two kinds of representations (π, V) of $\mathfrak{g}$, the adjoint representation $(\mathrm{ad}, \mathfrak{g})$ and irreducible representations $(\pi_\Lambda, L(\Lambda))$ with dominant integral highest weights $\Lambda\in\mathfrak{h}_{\boldsymbol{R}}^*$, and defined the spaces $H_m(\pi)$ of vectors of class C^m, $m=0,1,2,\cdots,\infty,\omega$. Then we showed that the action of the "analytic completion" $\mathfrak{k}_\omega$ of $\mathfrak{k}$ can be exponentiated and that the exponentials $\exp\pi(x)$, $x\in\mathfrak{k}_\omega$, leave each space of C^m-vectors invariant.

In this paper, we extend this result so that the action of $\mathfrak{k}_2$ on the space of C^1-vectors is exponentiated and that for each $m=0,1,2,\cdots$, the space of C^m-vectors is invariant under the exponentials of elements in $\mathfrak{k}_{m+2}$. ($\mathfrak{k}_2$ and $\mathfrak{k}_{m+2}$ will be defined in § 1)

§ 1. **Spaces of C^m-vectors.** The notations are the same as in [2]. Since the standard contravariant Hermitian form $(\cdot|\cdot)_0$ on $\mathfrak{g}$ is not positive definite on $\mathfrak{h}$ in general, we take another Hermitian form $(\cdot|\cdot)_1$ positive definite on the whole space $\mathfrak{g}$ as follows. Take a basis $\{h_i\}_i$ of $\mathfrak{h}_{\boldsymbol{R}}$ such that $(h_i|h_j)_0=\delta_{ij}$ or $-\delta_{ij}$ for any i, j. Let $(\cdot|\cdot)_1$ be the inner product on $\mathfrak{h}$ with respect to which $\{h_i\}_i$ is an orthonormal basis, and extend it to $\mathfrak{g}$ by

$$(x|y)_1=(x_0|y_0)_1+\sum(x_\alpha|y_\alpha)_0$$

$$\text{for } x=x_0+\sum x_\alpha,\ y=y_0+\sum y_\alpha\in\mathfrak{g}=\mathfrak{h}+\sum\mathfrak{g}^\alpha,$$

where all summations run over the root system Δ.

Let T be the bijective linear operator on $\mathfrak{g}$ such that $(x|y)_0=(x|Ty)_1$ for any $x, y\in\mathfrak{g}$. Then, as is easily verified, T is unitary and self-adjoint with respect to $(\cdot|\cdot)_1$, and so involutive.

Let $P(\pi)$ be the set of weights of (π, V) and put $\underline{V}=\prod_{\mu\in P(\pi)}V_\mu$, the direct product of V_μ's, where V_μ is the weight space of weight μ. Then, $\mathfrak{g}$ acts on $\underline{V}$ by

$$\pi(x)v=\left(\textstyle\sum_{\alpha+\nu=\mu}\pi(x_\alpha)v_\nu\right)_\mu$$

for $x=x_0+\sum x_\alpha\in\mathfrak{g}=\mathfrak{h}+\sum\mathfrak{g}^\alpha$, $v=(v_\mu)_\mu\in\underline{V}$.

Let $(\cdot|\cdot)_\pi$ be the standard inner product on V: $(\cdot|\cdot)_\pi=(\cdot|\cdot)_1$ for $\pi=\mathrm{ad}$ and $(\cdot|\cdot)_\pi=(\cdot|\cdot)_\Lambda$ in [2] for $\pi=\pi_\Lambda$. Further let $H(\pi)$ be the completion of the pre-Hilbert space $(V, (\cdot|\cdot)_\pi)$. Then $H(\pi)$ is regarded as a subspace of $\underline{V}$ by

$$H(\pi)=\{(v_\mu)_\mu \in \underline{V} \mid \sum_\mu (v_\mu \mid v_\mu)_\pi < +\infty\}.$$

In [3], we defined subspaces $H_m(\pi)$, the spaces of C^m-vectors, of $H(\pi)$ by $H_0(\pi)=H(\pi)$, and

$$H_m(\pi)=\{v \in H_{m-1}(\pi) \mid \pi(x)v \in H_{m-1}(\pi) \text{ for any } x \in \mathfrak{g}\}.$$

Then, each $H_m(\pi)$ is characterized by one arbitrarily fixed strictly dominant element in $\mathfrak{h}_{\boldsymbol{R}}$ as follows.

Proposition 1 [3, Theorem 3.2]. *Let $h_0 \in \mathfrak{h}_{\boldsymbol{R}}$ be a strictly dominant element, viz, an element such that $\alpha(h_0)>0$ for any positive root α. Then, it holds that for any $m=0,1,2,\cdots$,*

$$H_m(\pi)=\{v \in \underline{V} \mid \pi(h_0)^m v \in H(\pi)\}.$$

Thanks to this characterization, we can define for each $m=1,2,3,\cdots$, an inner product $(\cdot\mid\cdot)_{\pi,m}$ on $H_m(\pi)$ which provides $H_m(\pi)$ with a Hilbert space structure, and a continuous imbedding $H_m(\pi)\hookrightarrow H_{m-1}(\pi)$. The action of $\mathfrak{g}$ on V is extended, by continuity, to a bilinear map $H_m(\mathrm{ad})\times H_m(\pi)\ni(x,v)\mapsto\pi(x)v\in H_{m-1}(\pi)$. We write $[x,y]$, $x,y\in H_m(\pi)$, for $(\mathrm{ad}\,x)y$.

Let $\mathfrak{g}_m=H_m(\mathrm{ad})$ and $\mathfrak{k}_m$ be the closure of the unitary form $\mathfrak{k}$ in $\mathfrak{g}_m$.

§2. Negative spaces. To show the exponentiability of the actions of the completions of $\mathfrak{k}$, we need to introduce *negative spaces* $H_{-m}(\pi)$ as the duals of the spaces $H_m(\pi)$ of C^m-vectors.

Let $m=0,1,2,\cdots$, and $v\in H(\pi)$. Since the inclusion $H_m(\pi)\hookrightarrow H(\pi)$ is continuous, a continuous linear form F_v on $H_m(\pi)$ is defined by $F_v(u)=(u\mid v)_\pi$ for $u\in H_m(\pi)$. Let $\|v\|_{\pi,-m}$ be the norm of the linear form F_v, and $H_{-m}(\pi)$ the completion of $H(\pi)$ with respect to this norm.

We may regard canonically all the spaces $H_{-m}(\pi)$ as subspaces of $\underline{V}$ and we have a chain of Hilbert spaces spreading into two sides:

$$\underline{V}\supset\cdots\supset H_{-m-1}(\pi)\supset H_{-m}(\pi)\supset\cdots\supset H_{-1}(\pi)\supset H_0(\pi)\supset H_1(\pi)\supset$$
$$\supset\cdots\supset H_m(\pi)\supset H_{m+1}(\pi)\supset\cdots\supset V.$$

By definition of $H_{-m}(\pi)$, $(\cdot\mid\cdot)_\pi$ gives a non-degenerate sesquilinear pairing on $H_m(\pi)\times H_{-m}(\pi)$. Through this pairing, the action of $\mathfrak{g}_{m+1}$ on $H_{m+1}(\pi)$ is translated on $H_{-m}(\pi)$ as $(u\mid\pi(x)v)_\pi=((T_\pi\circ\pi(x^*)\circ T_\pi)u\mid v)_\pi$, for $x\in\mathfrak{g}_{m+1}$, $u\in H_{m+1}(\pi)$ and $v\in H_{-m}(\pi)$, where $T_\pi=T$ for $\pi=\mathrm{ad}$ and $T_\pi=$identity for $\pi=\pi_\Lambda$.

§3. Exponentials of $x\in\mathfrak{k}_m$. Here, we recall the following criterion in [4, Chapter IX] for the exponentiability of a closed operator on a Banach space.

Proposition 2 [4]. *Let $(X,\|\cdot\|)$ be a Banach space and B a closed operator on X with the dense domain $D\subset X$. Assume that B satisfies the following conditions: for sufficiently small $\varepsilon\in\boldsymbol{R}$,* i) *$1-\varepsilon B$ is surjective, and* ii) *there exists a positive constant c independent of ε such that for any $v\in D$,*

$$\|(1-\varepsilon B)v\|\geqq(1-c|\varepsilon|)\|v\|.$$

Then, there exists a unique strongly continuous 1-parameter group S_t, $t\in\boldsymbol{R}$, of bounded operators on X whose infinitesimal generator is equal to B. The operator norm is evaluated as $\|S_t\|\leqq e^{c|t|}$.

Now, we show that this criterion can be applied to the closure of $\pi(x)$,

$x \in \mathfrak{k}_{m+2}$, considered as an operator on $H_m(\pi)$ with the dense domain $H_{m+1}(\pi)$.

Put $|v|_{\pi,m}=\sum_{j=0}^{m}\|\pi(h_0)^j v\|_\pi$ for $v \in H_m(\pi)$. Then, the norm $|\cdot|_{\pi,m}$ is equivalent to the original one on $H_m(\pi)$, and $(H_m(\pi), |\cdot|_{\pi,m})$ is a Banach space. For this new norm, we have an important evaluation for the actions of elements in $\mathfrak{k}_{m+1}$ on $H_{m+1}(\pi)$ which fits the condition ii) in Proposition 2.

Proposition 3. *Let $x \in \mathfrak{k}_{m+1}$. Then, there exists a positive constant C dependent only on m and π such that for any $v \in H_{m+1}(\pi)$,*

$$|(1-\pi(x))v|_{\pi,m} \geqq (1-C|x|_{\mathrm{ad},m+1})|v|_{\pi,m}.$$

To examine the condition i) in Proposition 2, we need the following estimate for $\mathfrak{k}_{m+2}$-action on the negative space $H_{-m}(\pi)$.

Proposition 4. *Let $x \in \mathfrak{k}_{m+2}$. Then, there exist positive constants c and c' both dependent only on m and π such that for any $v \in H_{-m}(\pi)$,*

$$\|(1+\pi(x))v\|_{\pi,-m-1} \geqq c(1-c'\|x\|_{\mathrm{ad},m+2})\|v\|_{\pi,-m-1}.$$

Now let $x \in \mathfrak{k}_{m+2}$. By definition of the action of x on $H_{-m}(\pi)$, for any $\varepsilon \in \boldsymbol{R}$, $(1-\varepsilon\pi(x))H_{m+1}(\pi)$ is dense in $H_m(\pi)$ if and only if $1+\varepsilon\pi(x): H_{-m}(\pi) \to H_{-m-1}(\pi)$ is injective. Hence, by Proposition 4, if ε is sufficiently small, then $(1-\varepsilon\pi(x))H_{m+1}(\pi)$ is dense in $H_m(\pi)$. Let B be the closure of the operator $\pi(x)$ on $H_m(\pi)$ with the domain $H_{m+1}(\pi)$. By Proposition 3, the range of $1-\varepsilon B$ is equal to the closure of that of $1-\varepsilon\pi(x)$. And so $1-\varepsilon B$ is surjective, that is, B satisfies the condition i) in Proposition 2.

On the other hand, we see again from Proposition 3 that B satisfies also the condition ii), and we have

Theorem 5. *Let $m=0,1,2,\cdots$, and $x \in \mathfrak{k}_{m+2}$. Then, there exists a unique strongly continuous 1-parameter group $e^{t\pi(x)}=\exp t\pi(x)$, $t \in \boldsymbol{R}$, of bounded operators on $H_m(\pi)$ whose infinitesimal generator is equal to the closure of the operator $\pi(x)$ on $H_m(\pi)$ with domain $H_{m+1}(\pi)$. Moreover, the operator norm $|e^{\pi(x)}|_{\mathrm{op},\pi,m}$ with respect to $|\cdot|_{\pi,m}$ is evaluated as*

$$|e^{\pi(x)}|_{\mathrm{op},\pi,m} \leqq \exp(C|x|_{\mathrm{ad},m+1}),$$

where C is the same constant as in Proposition 3.

Naturally, if $x \in \mathfrak{k}_{m+2}$, the exponential $e^{\pi(x)}$ defined on $H(\pi)=H_0(\pi)$ coincides, by restriction, with $e^{\pi(x)}$ defined on $H_m(\pi)$.

§4. Properties of the exponential map. Here, we list up some properties of the map exp. First, we have the following continuity of exp.

Theorem 6. *Let $m=0,1,2,\cdots, x \in \mathfrak{k}_{m+2}$, $y \in \mathfrak{k}_{m+3}$, and $v \in H_{m+1}(\pi)$. Then, there holds for the constant C in Proposition 3*

$$|e^{\pi(x)}e^{\pi(y)}v-v|_{\pi,m} \leqq Ce^{C(|x|_{\mathrm{ad},m+1}+2|y|_{\mathrm{ad},m+2})}|x+y|_{\mathrm{ad},m+1}|v|_{\pi,m+1}.$$

In particular, the exponential map $\mathfrak{k}_{m+3} \ni z \mapsto e^{\pi(z)} \in \boldsymbol{B}(H_m(\pi))$, the space of all the bounded operators on $H_m(\pi)$, is strongly continuous with respect to the norm $\|\cdot\|_{\mathrm{ad},m+1}$ uniformly on any subset of $\mathfrak{k}_{m+3}$ which is bounded with respect to $\|\cdot\|_{\mathrm{ad},m+2}$.

Remark. It is shown in Theorem 5 that the exponentials $\exp \pi(x)$, $x \in \mathfrak{k}_{m+3}$, are contained in $\boldsymbol{B}(H_{m+1}(\pi))$. But, to imply the continuity of exp, we have to consider a weaker topology, the relative one from the strong operator topology on $\boldsymbol{B}(H_m(\pi))$, as is stated in Theorem 6.

By this continuity, the commutation relations of exponentials proved in [3] for the exponentials of the analytic completion $\mathfrak{k}_\omega$ of $\mathfrak{k}$ is generalized as follows.

Theorem 7. *Let $x \in \mathfrak{k}_4$, $y \in \mathfrak{g}_1$, and $z \in \mathfrak{k}_2$. Then, we have*

i) $e^{\pi(x)}\pi(y)e^{-\pi(x)} = \pi(e^{\mathrm{ad}\,x}y)$ *on* $H_1(\pi)$,

ii) $e^{\pi(x)}e^{\pi(z)}e^{-\pi(x)} = \exp \pi(e^{\mathrm{ad}\,x}z)$ *on* $H(\pi)$.

§5. Groups associated with $\mathfrak{k}_m$. Finally let K_m^π be the group of operators generated by exp $\pi(\mathfrak{k}_m)$. Thanks to Theorem 7, we have the adjoint action of K_m^π through K_m^{ad} as follows. For simplicity, we assume here that for any connected component S of the Dynkin diagram of the GCM A, there exists $i \in S$ such that $(\Lambda|\alpha_i)$ is not zero for the ith simple root α_i. Then,

Theorem 8. *Let $m = 4, 5, 6, \cdots$, and $\pi = \pi_\Lambda$. Under the above assumption for Λ, there exists a unique group homomorphism* $\mathrm{Ad} = \mathrm{Ad}_\pi$ *of K_m^π onto K_m^{ad} such that*

$$\mathrm{Ad}\,(e^{\pi(x)}) = e^{\mathrm{ad}\,x} \qquad \textit{for each } x \in \mathfrak{k}_m.$$

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