51. An Inequality of Chern Numbers of Bogomolov Type for Minimal Varieties

By Hajime TSUJI
Department of Mathematics, Tokyo Metropolitan University

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1988)

1. Introduction. Recently Mori proved the existence of minimal models for projective algebraic 3-folds ([5]) and it is expected that minimal models for projective manifolds exist in all dimension. Hence it is important to study minimal algebraic varieties.

In [6], the author proved semistability of tangent bundles of smooth minimal algebraic varieties with respect to the canonical polarization and an inequality of Chern numbers of Miyaoka-Yau type for them. In [1] Enoki generalized the author's result about semistability to the case of minimal Kähler spaces. Actually he proved semistability of tangent sheaves with respect to the canonical (weak) polarization for minimal Kähler spaces ([1, Theorem 1.1]).

Their methods are a little bit different because there are no ample divisors on Kähler spaces. Actually the author considered not only a perturbation of the canonical polarization but also a perturbation of the tangent bundle itself in [6] while Enoki used only a perturbation of the canonical polarization in [1]. This explains why Enoki's method does not yield an inequality of Chern numbers.

The purpose of this short note is to show that we can easily get an inequality of Chern numbers of Bogomolov type for minimal varieties just by combining the both methods.

Theorem 1. *Let X be a minimal algebraic variety of dimension $n(\geq 3)$ over $\mathbf{C}$ such that codimSing $X>2$. Then the inequality*

$$(-1)^n c_1^n(X) \leq (-1)^n \frac{2n}{n-1} c_1^{n-2}(X) c_2(X)$$

holds.

We note that this inequality is a corollary of the result in [4] in the case of $n=3$. In the case that K_X is ample, we may deduce the inequality by using [1, Theorem 1.1], [3] and [2]. Unfortunately this inequality does not seem to be sharp. This inequality can be proved by constructing a Kähler metric with a good control of the Ricci tensor. To get a sharp inequality (i.e. an inequality of Miyaoka-Yau type), it seems to be necessary to construct some (singular) Kähler-Einstein metric on X with a good control of the full curvature tensor.

The author would like to express his sincere thanks to Professor Enoki who kindly sent his preprint to the author.

This work was done while the author was at Harvard University supported by JSPS. He would like to express his thanks to JSPS for their generous financial support.

2. Proof of Theorem 1. Since the proof is almost an immediate consequence of the methods in [6] and [1], we shall give only a sketch of it. See [6] and [1] for the detail.

Let X be as in Theorem 1 and let $\pi: M \to X$ be a Hironaka resolution of X such that the exceptional set is a divisor with only simple normal crossings as singularities. Let H be a smooth ample divisor on M. Let us cover M by a finite number of open unit polydisks

$$\Delta_\alpha = \{(z_\alpha^1, \cdots, z_\alpha^n) \mid |z_\alpha^i| < 1, i=1, \cdots, n\}$$

such that if $\Delta_\alpha \cap H \neq \emptyset$, then

$$\Delta_\alpha \cap H = \{(z_\alpha^1, \cdots, z_\alpha^n) \mid z_\alpha^1 = 0\}.$$

Let V be a $\boldsymbol{Q}$-vector bundle on M defined by

$$V \mid \Delta_\alpha = \mathcal{O}_M(\partial/\partial z_\alpha^1, \cdots, \partial/\partial z_\alpha^n) \qquad \text{if } \Delta_\alpha \cap H = \emptyset,$$

$$V \mid \Delta_\alpha = \mathcal{O}_M((z_\alpha^1)^{1/m} \partial/\partial z_\alpha^1, \partial/\partial z_\alpha^2, \cdots, \partial/\partial z_\alpha^n) \qquad \text{if } \Delta_\alpha \cap H \neq \emptyset.$$

By an easy modification of the analysis in [6, Section 2] using the idea in [1, Section 3], we get a bounded nondegenerate almost Hermitian-Einstein metric which has a good curvature in L^1-sense on V (see Lemma 3.1 and Proposition 3.2 in [1]) with respect to a singular Kähler form on M which has a pole of order $1/m$ along H. Here we have used the singular Kähler metric as in [6] instead of the perturbation by a smooth Kähler form in [1], i.e. the perturbation by a smooth Kähler form $t\Phi$ in Proposition 3.2, (b) in [1] is not necessary because it has been absorbed in the perturbation of the tangent bundle. This is the key point of the proof. Then by using the same computation as in [1, Section 4], we see that V is $f^*(\pi^*(K_X) + (1/m)H)$-semistable on some finite Galois covering $f: N \to M$ as in [6]. Since $f^*(\pi^*(K_X) + (1/m)H)$ is ample, we get a Bogomolov inequality for V on N (cf. [2]). We note that $\pi^*(c_1(X))^{n-2} c_2(M) = c_1^{n-2}(X) c_2(X)$ holds by the assumption that *codimSing* $X > 2$. Dividing the both sides of the inequality by *deg* f and letting m tend to infinity as in [6], we complete the proof of Theorem 1.

For the cotangent sheaf Ω_X^1, by taking the dual of V and noting that $\pi^* \Omega_X^1 \subseteq \Omega_M^1$, we obtain the following theorem.

Theorem 2 (a part of [1, Corollary 1.2]). *Let X be a minimal algebraic variety over $\boldsymbol{C}$. Then Ω_X^1 is K_X-semistable.*

We can easily generalize Theorems 1 and 2 to the quasiprojective case as in Section 5 of [6] using the idea in [7].

References

[1] I. Enoki: Stability and negativity for tangent sheaves of minimal Kähler spaces. (preprint, Osaka University) (to appear in Proc. of Katata Symposium 87, Lect. Notes in Math., Springer).

[2] D. Gieseker: On a theorem of Bogomolov. Amer. J. of Math., **101**, 77–85 (1979).

[3] V. B. Mehta and A. Ramanathan: Semistable sheaves on projective varieties and their restriction to curves. Math. Ann., **258**, 213–224 (1982).
[4] Y. Miyaoka: The Chern classes and Kodaira dimension of a minimal variety. Algebraic Geometry, Sendai, 1985. Advanced Studies in Pure Math., **10** (ed. T. Oda), Kinokuniya, Tokyo; North-Holland, Amsterdam, 449–476 (1987).
[5] S. Mori: Flip theorem and the existence of minimal model for 3-folds. Jour. of A.M.S., **1**, 117–253 (1988).
[6] H. Tsuji: Stability of tangent bundles of minimal algebraic varieties (to appear in Topology).
[7] ——: An inequality of Chern numbers for open algebraic varieties. Math. Ann., **277**, 483–487 (1987).