47. *Azumaya Algebras Split by Real Closure*[†]

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1. Introduction. Let K be a commutative ring with identity element. For a (local) signature $\sigma: K \to GF(3)=\{0, \pm 1\}$, (which satisfies $\sigma(-1)=-1$, for any $a, b \in K$ $\sigma(ab)=\sigma(a)\sigma(b)$, and $\sigma(a)=0$ or $\sigma(a)=\sigma(b)$ imply $\sigma(a+b)=\sigma(b)$ cf. [4]), $P_\sigma=\{x \in K \mid \sigma(x)=0 \text{ or } 1\}$ satisfies the following conditions; $P_\sigma+P_\sigma \subseteq P_\sigma$, $P_\sigma \cdot P_\sigma \subseteq P_\sigma$, $P_\sigma \cup (-P_\sigma)=K$, and $\mathfrak{p}_\sigma=P_\sigma \cap (-P_\sigma)$ is a prime ideal of K. Then P_σ is an ordering in the meaning of [6]. Conversely, an ordering P of K defines a signature $\sigma_P: K \to GF(3)$; $\sigma_P(x)=0$ if $x \in P \cap (-P)$, $\sigma_P(x)=1$ if $x \in P$ and $x \notin -P$, and $\sigma_P(x)=-1$ if $x \in -P$ and $x \notin P$. Therefore, we can identify σ and P_σ, (or P and σ_P). By $\mathrm{Sig}(K)$, we denote the set $\{\sigma: K \to GF(3) \mid \text{signature on } K\}$ $(=\{P \mid \text{ordering on } K\})$. Let P_0 be an ordering on K. For the prime ideal $\mathfrak{p}_0=P_0 \cap (-P_0)$ of K, $(\bar{K}_0, \bar{P}_0)$ denotes the totally ordered quotient field of the totally ordered domain $(K/\mathfrak{p}_0, P_0/\mathfrak{p}_0)$, and R_0 the real closure of the totally ordered field $(\bar{K}_0, \bar{P}_0)$. Let A be a K-algebra with identity element such that A is a finitely generated projective K-module. Then, there are elements $a_1, a_2, \cdots, a_n \in A$ and $\psi_1, \psi_2, \cdots, \psi_n \in \mathrm{Hom}_K(A, K)$ such that $a=\sum_{i=1}^n \psi_i(a)a_i$ for all $a \in A$. The trace map $t_r: A \to K$; $a \rightsquigarrow \sum_{i=1}^n \psi_i(aa_i)$ defines a quadratic K-module (A, ρ) by $\rho(a)=\mathrm{tr}(a^2)$ for $a \in A$. If $L \supset K$ is a commutative Galois extension with a finite Galois group G, then $\mathrm{tr}(a)=t_G(a) := \sum_{\sigma \in G} \sigma(a)$ holds for all $a \in A$ (cf. [2]). Let A be an Azumaya K-algebra. We shall say A to be P_0-split, if $A \otimes_K R_0$ is a matrix ring over R_0. Furthermore, we shall say that A is *real split*, if A is P-split for all $P \in \mathrm{Sig}(K)$. By $B(K, P_0)$ and $B^r(K)$, we denote the subgroups $\{[A] \in B(K) \mid A: P_0\text{-split}\}$ and $\{[A] \in B(K) \mid A: \text{real split}\}$ of the Brauer group $B(K)$ of K, respectively. Then, $B^r(K)=\bigcap_{P \in \mathrm{Sig}(K)} B(K, P)$. Let $L \supseteq K$ be a commutative ring extension with common identity element. Then we put $\mathrm{Sig}_{P_0}(L/K) := \{P \in \mathrm{Sig}(L) \mid P \cap K=P_0\}$, and $Q(K) := \bigcap_{P \in \mathrm{Sig}(K)} P$. $Q(K|L)$ denotes the intersection of all P in $\mathrm{Sig}(K)$ such that $\mathrm{Sig}_P(L/K)=\emptyset$. A quadratic K-module (M, q) is said to be *positive semi-definite*, if $q(x)$ belongs to $Q(K)$ for all $x \in M$. In this paper, we prove the following theorem:

Theorem. *Let $L \supset K$ be a Galois extension of commutative rings with a finite Galois group G in the meaning of* [2]. *Then, the following assertions hold:*

1) *If the quadratic K-module (L, ρ) is positive semi-definite, then $B(L/K)(:=\{[A] \in B(K) \mid A \otimes_K L \sim L: A \text{ is split by } L\})$ is included in $B^r(K)$.*

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2) *If* $|G|$ *is odd, then* $B(L/K)\subseteq B^r(K)$.

3) *Suppose that* $G=\langle\sigma\rangle$ *is a cyclic group, and* $A=\Delta(f, L, \Phi, G)=\sum_i J_{\sigma^i}$ *is a generalized crossed product of* L *and* G *with any factor set* (cf. [3]). *Then, there is an* L-L-*isomorphism* $g: \otimes_L^n J_\sigma\to L$, *where* $n=|G|$ *and* $\otimes_L^n J_\sigma=J_\sigma\otimes_L J_\sigma\otimes_L\cdots\otimes_L J_\sigma$. A *is real split if and only if* $g(\otimes^n x)\in Q(K|L)$ *for all* $x\in J_\sigma$, *where* $\otimes^n x=x\otimes\cdots\otimes x\in\otimes^n J_\sigma$.

2. P-splitting Azumaya K-algebra. Let $P\in \mathrm{Sig}(K)$, $\mathfrak{p}=P\cap(-P)$, and let $(\bar{K}, \bar{P})$ be the totally ordered quotient field of totally ordered domain $(K/\mathfrak{p}, P/\mathfrak{p})$ and R a real closure of $(\bar{K}, \bar{P})$.

Definition. For an Azumaya K-algebra A, the trace map $\mathrm{tr}\otimes I_{\bar{K}}$: $A\otimes_K\bar{K}\to\bar{K}$; $a\otimes\bar{c}\rightsquigarrow \mathrm{tr}(a)\bar{c}$ defines a quadratic form $\rho\otimes I_{\bar{K}}$: $A\otimes_K\bar{K}\to\bar{K}$; $\alpha\rightsquigarrow \mathrm{tr}\otimes I_{\bar{K}}(\alpha^2)$. By $\mathrm{sgn}_{(\bar{K},\bar{P})}(A\otimes_K\bar{K}, \rho\otimes I_{\bar{K}})$, we denote the value of signature of the quadratic form $\rho\otimes I_{\bar{K}'}$, in the ordinary meaning, under the totally ordered field $(\bar{K}, \bar{P})$.

Lemma 1. *For any Azumaya K-algebra A,* $\mathrm{sgn}_{(\bar{K},\bar{P})}(A\otimes_k\bar{K}, \rho\otimes I_{\bar{K}})$ *is either* $\sqrt{[A\otimes_K\bar{K}:\bar{K}]}$ *or* $-\sqrt{[A\otimes_K\bar{K}:\bar{K}]}$, *hence we can define*

$\mathrm{sgn}_P(A):=\mathrm{sgn}_{(\bar{K},\bar{P})}(A\otimes_K\bar{K}, \rho\otimes I_{\bar{K}})/\sqrt{[A\otimes_K\bar{K}:\bar{K}]}$, *so* $\mathrm{sgn}_P(A)=\pm 1$.
Then, we have the following;

1) $\mathrm{sgn}_P(A)=+1$ *if and only if A is P-split.*

2) *If* $[A\otimes_K\bar{K}:\bar{K}]$ *is odd, then* $\mathrm{sgn}_P(A)=+1$.

3) $[B(K):B(K, P)]=2$.

Proof. Since R is a real closed field, $A\otimes_K R$ is isomorphic to either a matrix ring R_n over R or a matrix ring D_m over a quaternion R-algebra $D=R\oplus Ri\oplus Rj\oplus Rij$ with $i^2=j^2=-1$ and $ij=-ji$. (i) The case $A\otimes_K R\cong R_n$, $[A\otimes_K\bar{K}:\bar{K}]=n^2$: Let $\{e_{pq}|p, q=1, 2, \cdots, n\}$ be the matrix units of $A\otimes_K R$, and $X=\sum_{p,q}X_{pq}e_{pq}$ any element of $A\otimes_K R$ with $X_{pq}\in R$. Easily, we get

$$\mathrm{tr}\otimes I_R(X^2)=n\sum_{p,q}X_{pq}X_{qp}=n(X_{11}^2+X_{22}^2+\cdots+X_{nn}^2+2\sum_{p<q}X_{pq}X_{qp}).$$

By a regular linear transformation;

$$X_{pp}=Y_{pp};\qquad p=1,2,\cdots,n,$$
$$\text{for } p<q;\quad \begin{cases}X_{pq}=Y_{pq}+Y_{qp}\\ X_{qp}=Y_{pq}-Y_{qp},\end{cases}$$

we have $\rho\otimes I_R(X)=\mathrm{tr}\otimes I_R(X^2)=n\{Y_{11}^2+Y_{22}^2+\cdots+Y_{nn}^2+2\sum_{p<q}(Y_{pq}^2-Y_{qp}^2)\}$, hence $\mathrm{sgn}_{(\bar{K},\bar{P})}(A\otimes_K\bar{K}, \rho\otimes I_{\bar{K}})=\mathrm{sgn}_R(A\otimes_K R, \rho\otimes I_R)=n$. (ii) The case $A\otimes_K R\cong D_m$, $[A\otimes_K\bar{K}:\bar{K}]=4m^2$: For the matrix units $\{e'_{pq}|p, q=1, 2, \cdots, m\}$ of $A\otimes_K R$, any element $X\in A\otimes_K R$ is expressed as $X=\sum_{p,q}(W_{pq}+X_{pq}i+Y_{pq}j+Z_{pq}ij)e'_{pq}$ for $W_{pq}, X_{pq}, Y_{pq}, Z_{pq}\in R$. By the same computation as in (i), we get

$$\rho\otimes I_R(X)=4m\{\textstyle\sum_p(W_{pp}^2-X_{pp}^2-Y_{pp}^2-Z_{pp}^2)$$
$$+2\textstyle\sum_{p<q}(W_{pq}W_{qp}-X_{pq}X_{qp}-Y_{pq}Y_{qp}-Z_{pq}Z_{qp})\},$$

and $\mathrm{sgn}_{(\bar{K},\bar{P})}(A\otimes_K\bar{K}, \rho\otimes I_{\bar{K}})=\mathrm{sgn}_R(A\otimes_K R, \rho\otimes I_R)=-2m(=-n)$. Thus, 1) and 3) follow from the above results in two cases (i) and (ii). 2) If n is odd, then the case (ii) is impossible.

Lemma 2. *Let $L\supset K$ be a commutative Galois extension with a finite Galois group G, and suppose* $\mathrm{Sig}_P(L/K)\neq\emptyset$.

1) *If* $A=\Delta(f, L, \Phi, G)$ *is a generalized crossed product of* L *and* G *with any factor set* f *and* $\Phi: G\to \mathrm{Pic}_K(L)$, *then* $\mathrm{sgn}_P(A)=+1$.

2) $B(L/K)\subseteq B(K, P)$.

Proof. By [5]; Proposition 9, $\mathrm{Sig}_P(L/K)\neq\emptyset$ implies $L\otimes_K R=e_1R\oplus e_2R\oplus\cdots\oplus e_nR$, where $e_1, e_2, \cdots, e_n$ are orthogonal idempotents with $1\otimes 1=e_1+e_2+\cdots+e_n$. 1) If $\mathrm{sgn}_P(A)=-1$, then $A\otimes_K R$ becomes a matrix ring D_m over a quaternion R-algebra D, where $2m=n=|G|$. Since $L\otimes_K R$ is a maximal commutative subalgebra of $A\otimes_K R$, $A\otimes_K R$ has a left ideal decomposition $A\otimes_K R=A\otimes_K Re_1\oplus\cdots\oplus A\otimes_K Re_n$. The dimension of a minimal left ideal of D_m over D is equal to m, so we get $[(A\otimes_K R)e_i : R]\geqq 4m=2n$ $i=1, 2, \cdots, n$. However, we have $n^2=[A\otimes_K R : R]=\sum_{i=1}^n [A\otimes_K Re_i : R]\geqq 2n^2$, this is a contradiction. Hence, $\mathrm{sgn}_P(A)=+1$. 2) Let $[A]$ be any element in $B(L/K)$. By [1]; Theorem 5.7, there exists an Azumaya K-algebra A_0 such that A_0 has L as a maximal commutative subalgebra and $[A_0]=[A]$. By [4]; Proposition 3, there are a factor set f and a group homomorphism $\Phi: G\to\mathrm{Pic}_K(L)$ such that $A_0\cong\Delta(f, L, \Phi, G)$. Hence, by 1) we get $[A]=[A_0]\in B(K, P)$.

Let $L\supset K$ be a cyclic Galois extension of commutative rings with a Galois group $G=\langle\sigma\rangle$ of order n, and $A=\Delta(f, L, \Phi, G)$ a generalized crossed product of L and G with any factor set f and $\Phi: G\to\mathrm{Pic}_K(L)$. Then, A is expressed as $\sum_i \oplus J_{\sigma^i}$, where $J_{\sigma^i}=\otimes_L^i J_\sigma := J_\sigma\otimes_L J_\sigma\otimes\cdots\otimes_L J_\sigma$ (i times tensor product of J_σ over L), and there is an L-L-isomorphism $g: \otimes_L^n J_\sigma\to L$.

Lemma (cf. [5]; Proposition 9). *Let* $L\supset K$ *be a commutative Galois extension with a finite abelian Galois group* G, *and suppose* $\mathrm{Sig}_P(L/K)=\emptyset$. *For any element* $a\in K$, $a\in P$ *if and only if there exists an element* α *in* $L\otimes_K R$ *such that* $a\otimes 1=N_G(\alpha) := \prod_{\sigma\in G}\sigma(\alpha)$ *in* $\bar{K}=K\otimes_K\bar{K}$.

Proposition 3. *Let* $L, A=\Delta(f, L, \Phi, G)=\sum_i\oplus J_{\sigma^i}$ *and* $g: \otimes_L^n J_\sigma\to L$ *be as above. For any* $x\in J_\sigma$, $g(\otimes^n x)$ *is contained in* K. *Suppose* $\mathrm{Sig}_P(L/K)=\emptyset$. *Then,* $\mathrm{sgn}_P(A)=+1$ *if and only if* $g(\otimes^n x)$ *belongs to* P *for every* $x\in J_\sigma$.

Proof. From L-L-isomorphism $g: \otimes_L^n J_\sigma\to L$, it follows that there is an element u in $\otimes_L^n J_\sigma$ such that $g(u)=1$ and $\otimes_L^n J_\sigma=Lu=uL$. Let x be any element of J_σ. For any prime ideal $\mathfrak{p}$ of K, the localization $L_\mathfrak{p}=L\otimes_K K_\mathfrak{p}\supset K_\mathfrak{p}$ is also a Galois extension with Galois group G, and $A\otimes_K K_\mathfrak{p}=\sum_i\oplus J_{\sigma^i}\otimes_K K_\mathfrak{p}=\sum_i\oplus L_\mathfrak{p} u_\sigma^i$ is a free $L_\mathfrak{p}$-module with a free basis $u_\sigma, u_\sigma^2, \cdots, u_\sigma^n$. Elements $x\otimes 1\in J_\sigma\otimes_K K_\mathfrak{p}$ and $(\otimes^n x)\otimes 1$, $\otimes^n u_\sigma\in\otimes_L^n J_\sigma\otimes_K K_\mathfrak{p}$ are expressed as $x\otimes 1=\alpha u_\sigma$, $(\otimes^n x)\otimes 1=N_G(\alpha)\cdot\otimes^n u_\sigma$ and $\otimes^n u_\sigma=\beta u\otimes 1$ by $\alpha, \beta\in L_\mathfrak{p}$, and $N_G(\alpha)=\prod_{i=1}^n\sigma^i(\alpha)\in K_\mathfrak{p}$. However, $\otimes^n u_\sigma$ belongs to the center of $A_\mathfrak{p}$, hence $g\otimes I(\otimes^n u_\sigma)=\beta\in K_\mathfrak{p}$. Therefore, $g(\otimes^n x)\otimes 1$ belongs to $K_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of K, so we have $g(\otimes^n x)\in K$. Now, we suppose $\mathrm{Sig}_P(L/K)=\emptyset$. Considering a localization of A as above for the prime ideal $\mathfrak{p}=P\cap -P$ of K, it follows that $A\otimes_K R=\sum_i\oplus(L\otimes_K R)(\otimes^i u_\sigma\otimes 1)$ is a crossed product of $L\otimes_K R$ and $G=\langle\sigma\rangle$ with the factor set $g\otimes I(\otimes^n u_\sigma)=\beta\otimes 1$ ($\in K_\mathfrak{p}\otimes_K R$). As is well known, $A\otimes_K R\sim R$ if and only if $g\otimes I(\otimes^n u_\sigma)=\beta\otimes 1\in N_G(L\otimes_K R)$, that is, $g(\otimes^n x)\otimes 1\in N_G(L\otimes_K R)$ for every $x\in J_\sigma$. By the above lemma, we get that A is P-split, i.e. $\mathrm{sgn}_P(A)=+1$, if and only if $g(\otimes^n x)\in P$ for every $x\in J_\sigma$.

Corollary 4. *Let $L\supset K$ be a commutative Galois extension with a cyclic Galois group $G=\langle\sigma\rangle$ of order n. For any generalized crossed product $A=\Delta(f, L, \Phi, G)$ with an L-L-isomorphism $g:\otimes^n J_\sigma\to L$, A is P-split if and only if either $\mathrm{Sig}_P(L/K)\neq\emptyset$ or ($\mathrm{Sig}_P(L/K)=\emptyset$ and) $g(\otimes^n x)\in P$ for every $x\in J_\sigma$. Especially, if K is a field and $A=\Delta(L, G, a)=\sum_i \oplus L(u_\sigma)^i$ is a crossed product of L and $G=\langle\sigma\rangle$ with $(u_\sigma)^n=a\neq 0$ ($\in K$), then A is P-split if and only if either $\mathrm{Sig}_P(L/K)\neq\emptyset$ or $a>0$ under the ordering P on K.*

Remark (cf. [7]). Let K and L be fields such that $L\supset K$ a cyclic Galois extension with Galois group $G=\langle\sigma\rangle$ of order $2m$, and $A=\Delta(L, G, a)=L\oplus Lu\oplus\cdots\oplus Lu^{2m-1}$ a cyclic K-algebra with $u^{2m}=a$. When L_0 denotes the $\langle\sigma^m\rangle$-fixed subfield of L, then one can choose b in L_0 with $L=L_0(\sqrt{b})$. For the trace form ρ_{L_0} of L_0, we denote by $\rho_{L_0}b$ a quadratic form $\rho_{L_0}b: L_0\to K$; $x\leadsto \mathrm{tr}(bx^2)$. Then, the trace form (A, ρ_A) of A is expressed as follows;

$$\rho_A\approx\langle 4m\rangle\{\rho_{L_0}\perp a\rho_{L_0}\perp(\rho_{L_0}b)\perp -a(\rho_{L_0}b)\}\perp H,$$

where H is a hyperbolic space.

3. Proof of Theorem.

Lemma (cf. [5]; Theorem). *Let $L\supset K$ be a commutative Galois extension with a finite Galois group G.*

1) *The quadratic K-module (L, ρ) is positive semi-definite if and only if $\mathrm{Sig}_P(L/K)\neq\emptyset$ for all $P\in\mathrm{Sig}(K)$.*

2) *If $|G|$ is odd, then (L, ρ) is positive semi-definite.*

The proof of Theorem is obtained as follows; 1) From the above lemma and Lemma 2, it follows that $B(L/K)\subseteq B(K, P)$ for every $P\in\mathrm{Sig}(K)$, and so $B(L/K)\subseteq B^r(K)$. 2) is obtained by the above lemma and 1). 3) Suppose that $L\supset K$ is a commutative cyclic Galois extension with a Galois group G of order n, and that $A=\sum_i\oplus J_{\sigma^i}$ is a generalized crossed product of L and G with L-L-isomorphism $g:\otimes^n J_\sigma\to L$. By Corollary 4, A is real split, if and only if $g(\otimes^n x)$ belongs to P for all $x\in J_\sigma$ and every $P\in\mathrm{Sig}(K)$ with $\mathrm{Sig}_P(L/K)=\emptyset$, that is, $g(\otimes^n x)\in Q(K/L)$ for all $x\in J_\sigma$.

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