43. On Representations of Lie Superalgebras. II

By Hirotoshi FURUTSU
Department of Mathematics, College of Science and Technology,
Nihon University

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1988)

In this note we introduce a new method of constructing irreducible unitary representations (=IURs) of a classical Lie superalgebra of type A. Then we classify all the irreducible unitary representations of real forms of Lie superalgebra $A(1, 0)$. In the previous papers [2], [3], we define unitary representations of Lie superalgebras and introduce a general method of constructing irreducible representations of any simple Lie superalgebras. Moreover we classify and construct all the irreducible (unitary) representations of classical Lie superalgebra $\mathfrak{osp}(1,2)$. Further we gave similar results for real forms of the Lie superalgebra $\mathfrak{sl}(2,1)$ $(=A(1,0))$ exhaustively for the case where the even parts of representations are irreducible. There remains to study the case of non irreducible even parts.

1. New method. We have a Z-gradation $\mathfrak{g}_C=\mathfrak{g}_C^{-1}\oplus\mathfrak{g}_C^0\oplus\mathfrak{g}_C^{+1}$ with $\mathfrak{g}_C^0=\mathfrak{g}_{C,0}$ the even part, of Lie superalgebras $\mathfrak{g}_C=A(n,0)$ compatible with the Z_2-gradation $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ of a real form $\mathfrak{g}$ of $\mathfrak{g}_C$. A new method consists of the following. (i) First we study the weight distributions for IURs (π, V), and see in particular that any IUR is a highest (or lowest) weight representation because of its unitarity (see Proposition 1). (ii) Next we consider induced $\mathfrak{g}_C$-module $\bar{V}(\Lambda)=\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}_C}L(\Lambda)$. Here $\mathfrak{p}=\mathfrak{g}_C^0\oplus\mathfrak{g}_C^{+1}$, and $L(\Lambda)$ is an irreducible highest weight representation of $\mathfrak{g}_{C,0}$ with highest weight Λ considered as $\mathfrak{p}$-module by putting $\mathfrak{g}_C^{+1}$-action as trivial. Any irreducible representation $V(\Lambda)$ of $\mathfrak{g}_C$ with highest weight Λ is a quotient of $\bar{V}(\Lambda)$. (iii) Therefore we should determine the maximal submodule $I(\Lambda)$ of $\bar{V}(\Lambda)$ to get $V(\Lambda)\cong\bar{V}(\Lambda)/I(\Lambda)$.

2. Preliminaries. Denote by $M(p, q; K)$ the set of all matrices of type $p\times q$ with entries in a field K, and by $\mathfrak{g}_C$ the complex algebra $\mathfrak{sl}(n,1)=A(n-1,0)$. Here $\mathfrak{sl}(n,1)$ is realized in $M(n+1,n+1;C)$ as in [4, p. 29]. Let $\mathfrak{h}_C$ be a Cartan subalgebra of $\mathfrak{g}_C$ consisting of diagonal matrices, then $C=\sum_{1\leq i\leq n}E_{i,i}+nE_{n+1,n+1}$ is in $\mathfrak{h}_C$, where $E_{i,j}$ is an element of $M(n+1,n+1;C)$ with components 1 at (i,j) and 0 elsewhere. Real forms $\mathfrak{g}$ of $\mathfrak{g}_C=\mathfrak{sl}(n,1)$ are isomorphic, up to transition to their duals, to one of the following: (a) $\mathfrak{sl}(n,1;R)$; (b) $\mathfrak{su}(n,1;p,1)$ $([n+1/2]\leqq i\leqq n)$. Here $\mathfrak{sl}(n,1;R)=\mathfrak{sl}(n,1)\cap M(n+1,n+1;R)$, and $\mathfrak{su}(n,1;p,1)=\mathfrak{su}(n,1;p,1)_0\oplus\mathfrak{su}(n,1;p,1)_1$ with $\mathfrak{su}(n,1;p,1)_s=\{X\in\mathfrak{sl}(n,1)_s;\ J_pX+(-1)^s\cdot{}^t\bar{X}\bar{J}_p=0\}$ for $s=0,1$, where ${}^tX$ is the transposed matrix of X and $J_p=\mathrm{diag}(1,\cdots,1,-1,\cdots,-1,\sqrt{-1})$ with p-times 1 and $(n-p)$-times -1.

3. Weight distributions. For $\mathfrak{g}=\mathfrak{sl}(n,1;\boldsymbol{R})$, there exist no IURs except the trivial one. So we put $\mathfrak{g}=\mathfrak{su}(n,1;p,1)$. From the positive-definiteness condition for unitarity, we get

Proposition 1. *Let (π, V) be an IUR of a real Lie superalgebra $\mathfrak{su}(n,1;p,1)$. Then there are $\{\varepsilon_k\}_{1\leq k\leq n}$, $\varepsilon_k=\pm 1$, satisfying*

(1) $\varepsilon_1=\cdots=\varepsilon_p=-\varepsilon_{p+1}=\cdots=-\varepsilon_n$, *and*

(2) *any weight $\rho\in\mathfrak{h}_C^*$ of V satisfies*

$$\varepsilon_k\rho(H_k)\geqq 0 \quad for \quad 1\leqq k\leqq n \quad with \quad H_k=E_{k,k}+E_{n+1,n+1}\in\mathfrak{h}_C.$$

In particular, any IUR of $\mathfrak{g}$ must be a highest or lowest weight module.

4. $\boldsymbol{Z}$-gradation. Let $\mathfrak{g}_C=\mathfrak{sl}(n,1)$ and $C'=(1-n)^{-1}C\in\mathfrak{h}_C$, then $\mathfrak{g}_C$ is decomposed into C'-eigenspaces as $\mathfrak{g}_C=\mathfrak{g}_C^{-1}\oplus\mathfrak{g}_C^0\oplus\mathfrak{g}_C^{+1}$ for which the even part $\mathfrak{g}_{C,0}=\mathfrak{g}_C^0$ and the odd part $\mathfrak{g}_{C,1}=\mathfrak{g}_C^{-1}\oplus\mathfrak{g}_C^{+1}$. Thus $\mathfrak{g}_C$ becomes a $\boldsymbol{Z}$-graded algebra. The universal enveloping algebra $\mathcal{U}(\mathfrak{g}_C^{-1})$ is decomposed into C'-eigenspaces as $\mathcal{U}(\mathfrak{g}_C^{-1})=\oplus_{0\leq k\leq n}\mathcal{U}(-k)$, where the C'-eigenvalue of $\mathcal{U}(-k)$ is $-k$.

5. Induced highest weight modules $\bar{V}(\Lambda)$. Take a subalgebra $\mathfrak{p}=\mathfrak{g}_C^0\oplus\mathfrak{g}_C^{+1}$ of $\mathfrak{g}_C$. For $\Lambda\in\mathfrak{h}_C^*$, denote by $L(\Lambda)$ (resp. $V(\Lambda)$) the irreducible highest weight representation of $\mathfrak{g}_C^0$ (resp. $\mathfrak{g}_C$) with highest weight Λ. We define a $\mathfrak{g}_C^{+1}$-action on $L(\Lambda)$ by $\zeta v=0$ ($\zeta\in\mathfrak{g}_C^{+1}$, $v\in L(\Lambda)$), then $L(\Lambda)$ becomes a p-module. Now define a $\mathfrak{g}_C$-module $\bar{V}(\Lambda)$ as in [5] by $\bar{V}(\Lambda)\equiv\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}_C}L(\Lambda)$. Then $\bar{V}(\Lambda)\cong\mathcal{U}(\mathfrak{g}_C^{-1})\otimes_C L(\Lambda)$ is decomposed into C'-eigenspaces as $\bar{V}(\Lambda)=\oplus_{0\leq k\leq n}\bar{V}_{-k}$, where $\bar{V}_{-k}=\mathcal{U}(-k)L(\Lambda)$ has eigenvalue $\Lambda(C')-k$. And we get the following criterion of irreducibility, which is first obtained by Kac [5] for $L(\Lambda)$ with $\dim L(\Lambda)<\infty$.

Proposition 2. *$\bar{V}(\Lambda)$ is irreducible if and only if*

$$\prod_{1\leq k\leq n}\{\Lambda(H_k)+n-k\}\neq 0.$$

6. Method of constructing $V(\Lambda)$ from $\bar{V}(\Lambda)$. Step 1: First we decompose each $\bar{V}_{-k}=\mathcal{U}(-k)L(\Lambda)$ into irreducibles of $\mathfrak{g}_0$, or determine its subquotient structure. Step 2: Check the $\mathfrak{g}_C^{-1}$-action on each component, that is, decompose $\mathfrak{g}_C^{-1}V_a$ into irreducible $\mathfrak{g}_0$-modules for each $\mathfrak{g}_0$-irreducible component V_a of $\bar{V}_{-k}$. (This decomposition is independent of the value $\Lambda(C)$.) Step 3: $\mathfrak{g}_C^{+1}V_a$ depends on the value of $\Lambda(C)$. So we calculate its structure case by case. Step 4: Finally, from Steps 2 and 3, we get the maximal submodule $I(\Lambda)$ and obtain $V(\Lambda)=\bar{V}(\Lambda)/I(\Lambda)$.

7. Classification of IURs for $\mathfrak{g}=\mathfrak{su}(2,1;2,1)$. Let $\mathfrak{g}_C=\mathfrak{sl}(2,1)$, and α,β be simple roots of $(\mathfrak{g}_C,\mathfrak{h}_C)$ given as $\alpha(H)=2$, $\alpha(C)=0$; $\beta(H)=-1$, $\beta(C)=-1$. Here $\{H=E_{1,1}-E_{2,2}, C\}$ is a basis of $\mathfrak{h}_C$. Another positive root is $\gamma=\alpha+\beta$, and we put $\delta=\beta+\gamma$. Then we get

Theorem 3. (1) *Any irreducible unitary representation V of Lie superalgebra $\mathfrak{su}(2,1;2,1)$ is a highest or lowest weight representation. If V is a highest weight IUR, then V is isomorphic to one of the representations $V(\Lambda)$ for which $\Lambda(H)$ is a non-negative integer and $\Lambda(C)$ is a real number with $\Lambda(C)\leqq-\Lambda(H)-2$ or $\Lambda(H)\leqq\Lambda(C)$.*

(2) *As $\mathfrak{g}_0$-modules, the above $V(\Lambda)$ is decomposed as follows:*

(i) $V(\Lambda)=L(\Lambda)$ *for* $\Lambda(C)=\Lambda(H)=0$,

(ii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\gamma)$ *for* $\Lambda(C)=\Lambda(H)\geqq 1$,
(iii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)$ *for* $\Lambda(C)=-\Lambda(H)-2$,
(iv) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\delta)$ *for* $\Lambda(H)=0$ *and* $\Lambda(C)<-2,\ 0<\Lambda(C)$,
(v) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\gamma)\oplus L(\Lambda-\delta)$ *otherwise.*
Here the even part of $V(\Lambda)$ *consists of* $L(\Lambda)$ *and* $L(\Lambda-\delta)$.

8. Classification of IURs for $\mathfrak{g}=\mathfrak{su}(2,1;1,1)$. Let H, C, α, β, γ and δ be as above.

Theorem 4. (1) *Any irreducible unitary representation* V *of Lie superalgebra* $\mathfrak{g}=\mathfrak{su}(2,1;1,1)$ *is a highest or lowest weight representation. If* V *is a highest weight IUR, then* V *is isomorphic to one of the representations* $V(\Lambda)$ *for which* $\Lambda(H)$ *is a non-positive integer and* $\Lambda(C)$ *is a real number with* $\Lambda(H)\leqq\Lambda(C)\leqq-\Lambda(H)-2$ *or* $\Lambda(H)=\Lambda(C)=0$.

(2) *As a* $\mathfrak{g}_0$*-module, the above* $V(\Lambda)$ *is decomposed into* $\mathfrak{g}_0$*-irreducible components as follows:*
(i) $V(\Lambda)=L(\Lambda)$ *for* $\Lambda(C)=\Lambda(H)=0$,
(ii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\gamma)$ *for* $\Lambda(C)=\Lambda(H)\leqq-1$,
(iii) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)$ *for* $\Lambda(C)=-\Lambda(H)-2\geqq 0$,
(iv) $V(\Lambda)=L(\Lambda)\oplus L(\Lambda-\beta)\oplus L(\Lambda-\gamma)\oplus L(\Lambda-\delta)$ *otherwise.*

Thus we classify all the IURs of all the real forms of the Lie superalgebra $\mathfrak{sl}(2,1)$.

9. Realization of IURs. Realizations of the above IURs are given in [1] and [3]. Here we pick up the case (iv) in Theorem 4 as an example. In this case $\mathfrak{g}_0\cong\mathfrak{u}(1,1)$, $\ell=-\Lambda(H)$ is a positive integer $\geqq 2$ and $m=\Lambda(C)$ is a real number with $-\ell<m<\ell-2$. Let $v_1^0\in L(\Lambda)$ be a unit highest vector of $V(\Lambda)$, and $\{v_k^0\}_{k\in N}$ be a standard orthonormal basis of $L(\Lambda)$ given inductively by

$$\sqrt{(k+\ell-1)k}\cdot v_{k+1}^0=E_{2,1}v_k^0 \qquad \text{for } k\in N=\{1,2,3,\cdots\}.$$

Next let $\{v_k^\delta\}_{k\in N}$ be a standard orthonormal basis of $L(\Lambda-\delta)$ determined by

$$\sqrt{(\ell+m)(\ell-m-2)}\cdot v_k^\delta=2\cdot E_{3,1}E_{3,2}v_k^0 \qquad \text{for } k\in N.$$

We define standard orthonormal bases $\{v_k^\beta\}_{k\in N}$ and $\{v_k^\gamma\}_{k\in N}$ of $L(\Lambda-\beta)$ and $L(\Lambda-\gamma)$ respectively by

$$\sqrt{(\ell-1)(\ell+m)}\cdot v_k^\beta=\sqrt{2}\,(\sqrt{\ell+k-2}\cdot E_{3,2}v_k^0-\sqrt{k-1}\cdot E_{3,1}v_{k-1}^0),$$
$$\sqrt{(\ell-1)(\ell-m-2)}\cdot v_k^\gamma=\sqrt{2}\,(\sqrt{k}\cdot E_{3,2}v_{k+1}^0-\sqrt{\ell+k-1}\cdot E_{3,1}v_k^0).$$

We write the operator $\zeta\in\mathfrak{g}_{1,C}$ in the form of blockwise matrix of operators $(D_{j,k})_{j,k=0,\beta,\gamma,\delta}$, where $D_{j,k}: L(\Lambda-k)\to L(\Lambda-j)$. Then $\zeta=(D_{j,k})$ is of the following form respectively depending on $\zeta\in\mathfrak{g}_C^{+1}$ or $\zeta\in\mathfrak{g}_C^{-1}$:

$$(D_{j,k})_{j,k=0,\beta,\gamma,\delta}=\begin{pmatrix}0&*&*&0\\0&0&0&*\\0&0&0&*\\0&0&0&0\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}0&0&0&0\\ *&0&0&0\\ *&0&0&0\\0&*&*&0\end{pmatrix}.$$

And the action of $\mathfrak{g}_{C,1}$ is respectively given as follows:

For $\mathfrak{g}_C^{+1}$: $D_{0,\beta}$ & $D_{0,\gamma}$: $E_{1,3}v_k^\beta=-\tilde{C}_-a_{k-1}v_{k-1}^0,\qquad E_{1,3}v_k^\gamma=\tilde{C}_+\tilde{b}_kv_k^0,$
$D_{\beta,\delta}$ & $D_{\gamma,\delta}$: $E_{1,3}v_k^\delta=-\tilde{C}_+\tilde{b}_{k-1}v_k^\beta-\tilde{C}_-a_{k-1}v_{k-1}^\gamma,$

$$\begin{array}{lll} & D_{0,\beta}\ \&\ D_{0,\gamma} : & E_{2,3}v_k^\beta=\tilde{C}_-\tilde{b}_{k-1}v_k^0, \qquad E_{2,3}v_k^\gamma=-\tilde{C}_+a_kv_{k+1}^0, \\ & D_{\beta,\delta}\ \&\ D_{\gamma,\delta} : & E_{2,3}v_k^\delta=\tilde{C}_+a_kv_{k+1}^\beta+\tilde{C}_-\tilde{b}_kv_k^\gamma, \\ \text{For } \mathfrak{g}_{\bar{c}}^{-1}: & D_{\beta,0}\ \&\ D_{\gamma,0} : & E_{3,1}v_k^0=\tilde{C}_-a_kv_{k+1}^\beta+\tilde{C}_+\tilde{b}_kv_k^\gamma, \\ & D_{\delta,\beta}\ \&\ D_{\delta,\gamma} : & E_{3,1}v_k^\beta=\tilde{C}_+\tilde{b}_{k-1}v_k^\delta, \qquad E_{3,1}v_k^\gamma=\tilde{C}_-a_kv_{k+1}^\delta, \\ & D_{\beta,0}\ \&\ D_{\gamma,0} : & E_{3,2}v_k^0=\tilde{C}_-\tilde{b}_{k-1}v_k^\beta-\tilde{C}_+a_{k-1}v_{k-1}^\gamma, \\ & D_{\delta,\beta}\ \&\ D_{\delta,\gamma} : & E_{3,2}v_k^\beta=\tilde{C}_+a_{k-1}v_{k+1}^\delta, \qquad E_{3,2}v_k^\gamma=\tilde{C}_-\tilde{b}_kv_k^\delta, \end{array}$$

where $a_k=\sqrt{k}$, $\tilde{b}_k=\sqrt{\ell+k-1}$ and $\tilde{C}_\pm=\{(\ell-1)\mp(m+1)\}^{1/2}\{2(\ell-1)\}^{-1/2}$.

References

[1] H. Furutsu: Representations of Lie superalgebras. II. Unitary representations of Lie superalgebras of type $A(n, 0)$ (preprint).

[2] H. Furutsu and T. Hirai: Proc. Japan Acad., **63A**, 235–238 (1987).

[3] ——: Representations of Lie superalgebras. I. Extensions of representations of the even part (to appear in J. Math. Kyoto Univ.).

[4] V. G. Kac: Lie superalgebras. Adv. in Math., **26**, 8–96 (1977).

[5] ——: Lecture Notes in Math., vol. 676, Springer-Verlag, pp. 597–626 (1978).