42. On the Initial Value Problem for the Heat Convection Equation of Boussinesq Approximation in a Time-dependent Domain

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Introduction and results. Let K be a compact set in R^m ($m=2$ or 3) with smooth boundary ∂K. Let $\Gamma(t)$ be a simple closed surface in R^3 (or curve in R^2) such that K is contained in the interior of the region surrounded by $\Gamma(t)$. The time-dependent space domain $\Omega(t)$ is a bounded set in R^m whose boundary $\partial\Omega(t)$ consists of two components, i.e.

$$\partial\Omega(t)=\partial K\cup\Gamma(t).$$

Such domains $\Omega(t)$ $(0\leqq t\leqq T)$ generate a non-cylindrical domain $\hat{\Omega}=\cup_{0\leqq t\leqq T}\cdot\Omega(t)\times\{t\}$, where we consider the following initial value problem for the heat convection equation of Boussinesq approximation:

$$(1)\quad \begin{cases} u_t+(u\cdot\nabla)u=-\dfrac{\nabla p}{\rho}+\{1-\alpha(\theta-T_0)\}g+\nu\Delta u & \text{in } \hat{\Omega},\\ \operatorname{div} u=0 & \text{in } \hat{\Omega},\\ \theta_t+(u\cdot\nabla)\theta=\kappa\Delta\theta & \text{in } \hat{\Omega}, \end{cases}$$

$$(2)\quad u|_{\partial\Omega(t)}=\beta(x,t),\quad \theta|_{\partial K}=T_0>0,\quad \theta|_{\Gamma(t)}=0 \quad \text{for any } t\in[0,T],$$

$$(3)\quad u|_{t=0}=a,\qquad \theta|_{t=0}=h \qquad \text{in } \Omega(0),$$

where $u=u(x,t)$ is the velocity field, $p=p(x,t)$ is the pressure and $\theta=\theta(x,t)$ is the temperature; ν,κ,α,ρ and $g=g(x)$ are the kinematic viscosity, the thermal conductivity, the coefficient of volume expansion, the density at $\theta=T_0$ and the gravitational vector, respectively. According to Boussinesq approximation, ρ is a fixed constant. The differential operators Δ and ∇ mean those for x variables only. Concerning the Navier-Stokes equation, Fujita-Sauer [1], Ôtani-Yamada [6], Inoue-Wakimoto [2] and H. Morimoto [5] studied the initial value problem or the time periodic problem in some time-dependent domains. As for the stationary problem for the heat convection equation, we refer to, for instance, P.H. Rabinowitz [7] and D.H. Sattinger [8]. We note, as a physical example, the convection of the earth's mantle which may occur in the interior of the earth.

We make some simplifying assumptions on $\beta(x,t)$ and $\Omega(t)$.

(A1) $\beta\equiv 0$. (This may not be physically realistic.)

(A2) There exists an open ball B_1 such that $\overline{\Omega(t)}\subset B_1$.

(A3) $\Gamma(t)$ and ∂K are smooth (say, of class C^3). Also $\Gamma(t)\times\{t\}$ $(0<t<T)$ changes smoothly (say, of class C^4) with respect to t. (Namely, the domain $\hat{\Gamma}=\cup_{0<t<T}\Gamma(t)\times\{t\}$ has the same properties as those in [1] and [6].)

(A4) $g(x)$ is a bounded and continuous vector function in $R^m\backslash\operatorname{int} K$.

Our main results are as follows. (The definition of weak solutions, strong solutions and the function spaces are to be given in the next section.)

Theorem 1. *Assume* (A1)–(A4). *If* $a \in H_\sigma(\Omega(0))$ *and* $h \in L^2(\Omega(0))$, *then there exists a weak solution of* (1)–(3) *for any time interval* $[0, T]$.

Theorem 2. *Under the same assumptions of Theorem* 1, *if* $a \in H^1_\sigma(\Omega(0))$, $h \in W^1_2(\Omega(0))$, $h|_{\partial K} = T_0$ *and* $h|_{\Gamma(0)} = 0$, *then there is a positive number* τ_0 *depending on* a, h *and* T_0 *such that the initial value problem* (1)–(3) *has a unique strong solution on* $[0, \tau_0]$.

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Notations and formulation. For a bounded domain Ω in R^m with smooth boundary $\partial\Omega$, we write $\|u\|_\Omega$ or simply $\|u\|$ instead of $\|u\|_{L^2(\Omega)}$. The inner product in $L^2(\Omega)$ is denoted by $(u, v)_{L^2(\Omega)}$, $(u, v)_\Omega$ or (u, v). The solenoidal function spaces are defined as usual:

$D_\sigma(\Omega) = \{\varphi \in C_0^\infty(\Omega)\,;\ \mathrm{div}\,\varphi = 0\}$,

$H_\sigma(\Omega) =$ the completion of $D_\sigma(\Omega)$ under the $L^2(\Omega)$-norm,

$H^1_\sigma(\Omega) =$ the completion of $D_\sigma(\Omega)$ under the $W^1_2(\Omega)$-norm.

For the time-dependent domain $\hat{\Omega} = \bigcup_{0 \le t \le T} \Omega(t) \times \{t\}$, described in the preceding section, we put

$\hat{D}_\sigma(\hat{\Omega}) = \{\varphi \in C_0^\infty(\hat{\Omega})\,;\ \mathrm{div}\,\varphi = 0\}$,

$\hat{H}^1_\sigma(\hat{\Omega}) =$ the completion of $\hat{D}_\sigma(\hat{\Omega})$ under the norm $\nu_\sigma(\cdot)$,

where $\nu_\sigma(u) = \|\nabla u\|_{\hat{\Omega}}$;

$\hat{D}(\hat{\Omega}) = \{\psi \in C^\infty(\widehat{\Omega(t) \cup \partial K})\,;\ \mathrm{supp} \subset \widehat{\Omega(t) \cup \partial K}$ and $\psi = 0$ on $\partial K\}$,

$\hat{H}^1(\hat{\Omega}) =$ the completion of $\hat{D}(\hat{\Omega})$ under the norm $\nu(\cdot)$,

where $\nu(u) = \|\nabla u\|_{\hat{\Omega}}$ and $\widehat{\Omega(t) \cup \partial K} = \bigcup_{0 \le t \le T} (\Omega(t) \cup \partial K) \times \{t\}$.

Moreover,

$\hat{\mathcal{D}}_\sigma(\hat{\Omega}) = \{\varphi \in \hat{D}_\sigma(\hat{\Omega})\,;\ \varphi = 0 \text{ at } t = T\}$,

$\hat{\mathcal{D}}(\hat{\Omega}) = \{\psi \in \hat{D}(\hat{\Omega})\,;\ \psi = 0 \text{ at } t = T\}$,

$\mathcal{U}(\hat{\Omega}) = \{\varphi \in \hat{H}^1_\sigma(\hat{\Omega})\,;\ \operatorname*{ess.sup.}_{0 \le t \le T} \|\varphi(t)\|_{L^2(\Omega(t))} < +\infty\}$,

$\mathcal{T}(\hat{\Omega}) = \{\psi \in \hat{H}^1(\hat{\Omega})\,;\ \operatorname*{ess.sup.}_{0 \le t \le T} \|\psi(t)\|_{L^2(\Omega(t))} < +\infty\}$.

We introduce an auxiliary function $\bar{\theta}(x, t)$ solving

$$(4)\qquad \begin{cases} \theta_t = \Delta\theta & \text{in } \hat{\Omega}, \\ \theta|_{\partial K} = T_0,\ \theta|_{\Gamma(t)} = 0 & \text{for any } t \in [0, T], \\ \theta|_{t=0} = \eta(x) & \text{in } \Omega(0), \end{cases}$$

where $\eta(x)$ satisfies $\Delta\eta = 0$ in $\Omega(0)$ with $\eta|_{\partial K} = T_0$ and $\eta|_{\Gamma(0)} = 0$.

Under these preparations we can define the weak solution of (1)–(3).

Definition 1. $U = {}^t(u, \theta)$ defined in $\hat{\Omega}$ is a weak solution of (1)–(3) if the following (i) and (ii) are satisfied:

(i) ${}^t(u, \theta - \bar{\theta}) \in \mathcal{U}(\hat{\Omega}) \times \mathcal{T}(\hat{\Omega})$.

(ii) For all $\Phi = {}^t(\varphi, \psi) \in \hat{\mathcal{D}}_\sigma(\hat{\Omega}) \times \hat{\mathcal{D}}(\hat{\Omega})$ the equality

$$(5)\qquad \int_0^T \{(U, \Phi_t) + (U, (u \cdot \nabla)\Phi + \nu(u, \Delta\varphi) + \kappa(\theta, \Delta\psi) + ((1 - \alpha(\theta - T_0))g, \varphi)\}dt$$
$$= \int_0^T \int_{\partial K} T_0 \frac{\partial \psi}{\partial n}\, ds dt - (A, \Phi(0))$$

holds, where $A={}^t(a, h)$.

We will now define the strong solution of (1)–(3). First of all, we consider the following proper lower semi-continuous functions and subdifferential operators:

$$(6)\qquad \varphi_B(U)=\begin{cases}\dfrac{1}{2}\displaystyle\int_B(\nu|\nabla u|^2+\kappa|\nabla\theta|^2)dx & \text{if } U\in H^1_\sigma(B)\times \mathring{W}^1_2(B),\\ +\infty & \text{if } U\in (H_\sigma(B)\times L^2(B))\backslash(H^1_\sigma(B)\times\mathring{W}^1_2(B)),\end{cases}$$

$$(7)\qquad \partial\varphi_B(U)={}^t(A_\sigma(B)u, -\kappa\Delta\theta)=A(B)U,$$

where $B=B_1\backslash K$, $A_\sigma(B)=-\nu P_\sigma(B)\Delta$ and $P_\sigma(B)$ is the orthogonal projection from $L^2(B)$ onto $H_\sigma(B)$. It is known that $D(A(B))$, the domain of the operator $A(B)$, is equal to $(W^2_2(B)\cap H^1_\sigma(B))\times(W^2_2(B)\cap\mathring{W}^1_2(B))$. We next define a closed convex set $K(t)$ of $H_\sigma(B)\times L^2(B)$ by

$$K(t)=\{U\in H_\sigma(B)\times L^2(B)\,;\ U=0 \text{ a.e. in } B\backslash\Omega(t)\}$$

for each $t\in[0,T]$ and write its indicator function by $I_{K(t)}$, that is, $I_{K(t)}(U)=0$ if $U\in K(t)$ and $I_{K(t)}(U)=+\infty$ if $U\in(H_\sigma(B)\times L^2(B))\backslash K(t)$. Here we define another p.l.s.c. function

$$(8)\qquad \varphi^t(U)=\varphi_B(U)+I_{K(t)}(U)\qquad \text{for each } t\in[0,T].$$

We consider the subdifferential operator $\partial\varphi^t$. It holds that $D(\partial\varphi^t)=\{U\in H_\sigma(B)\times L^2(B)\,;\ U|_{\Omega(t)}\in(W^2_2(\Omega(t))\cap H^1_\sigma(\Omega(t)))\times(W^2_2(\Omega(t))\cap\mathring{W}^1_2(\Omega(t))),\ U|_{B\backslash\Omega(t)}=0\}$ and $\partial\varphi^t(U)=\{f\in H_\sigma(B)\times L^2(B)\,;\ P(\Omega(t))f|_{\Omega(t)}=A(\Omega(t))U|_{\Omega(t)}\}$ where $P(\Omega(t))={}^t(P_\sigma(\Omega(t)), 1_{\Omega(t)})$. (See [6] and [9].) Then we can reduce the initial value problem (1)–(3) to the one for the following abstract heat convection equation (AHC) in $H_\sigma(B)\times L^2(B)$:

$$\text{(AHC)}\qquad \frac{dV}{dt}+\partial\varphi^t(V(t))+F(t)V(t)+M(t)V(t)\ni P(B)f(t),\qquad t\in[0,T],$$

where $V={}^t(v,\theta)$, $F(t)V(t)={}^t(P_\sigma(B)(v\cdot\nabla)v,\ (v\cdot\nabla)\theta)$, $M(t)V(t)={}^t(P_\sigma(B)\alpha\theta g,\ (v\cdot\nabla)\bar\theta)$, $f={}^t(f_1,f_2)={}^t((1-\alpha(\bar\theta-T_0))g, 0)$ and $P(B)={}^t(P_\sigma(B), 1_B)$. (See [6] and [9].)

We define the strong solution of (AHC) as follows.

Definition 2. Let $V:[0,S]\to H_\sigma(B)\times L^2(B)$, $S\in(0,T]$. Then V is called a strong solution of the initial value problem for (AHC) on $[0,S]$ if it satisfies the following properties (i), (ii) and (iii).

(i) $V\in C([0,S]\,;H_\sigma(B)\times L^2(B))$ and $dV/dt\in L^2(0,S\,;\ H_\sigma(B)\times L^2(B))$.

(ii) $V(t)\in D(\partial\varphi^t)$ for a.e. $t\in[0,S]$ and there is a function $G={}^t(g_1,g_2)\in L^2(0,S\,;H_\sigma(B)\times L^2(B))$ such that $G(t)\in\partial\varphi^t(V(t))$ and

$$\frac{dV}{dt}+G(t)+F(t)V(t)+M(t)V(t)=P(B)f(t)$$

hold for a.e. $t\in[0,S]$.

(iii) $V(0)={}^t(\tilde a, \tilde h-\tilde{\bar\theta}(0))$ holds in $H_\sigma(B)\times L^2(B)$ where $\tilde a$, $\tilde h$ and $\tilde{\bar\theta}$ mean the natural extension of a, h and $\bar\theta$, respectively.

Remark 1. Let V be a strong solution of (AHC). Then we can show that $U=V|_{\hat\Omega}+{}^t(0,\bar\theta)$ actually satisfies the heat convection equation for a.e. $t\in[0,S]$.

Outline of the proofs. Theorem 1 is proven by the method of [1], [4] and [5]. We employ the penalty and the Galerkin's approximation.

Theorem 2 is proven by an iteration. To show the convergence of the iterated sequence, the following is important:

Lemma 1. *Let* $U: [0, T] \to H_\sigma(B) \times L^2(B)$ *and* $\varphi^t(U(\cdot)): [0, T] \to [0, +\infty)$ *be absolutely continuous on* $[0, T]$. *Let* $\mathcal{L} \equiv \{t \in (0, T)\,;\, dU/dt,\ d\varphi^t(U(t))/dt$ *exist and* $U(t) \in D(\partial\varphi^t)\}$. *Then, there exist positive constants* C_1 *and* C_2 *such that*

$$(9)\qquad \left| \frac{d}{dt}\varphi^t(U(t)) - \left(G, \frac{d}{dt}U(t)\right)_{L^2(B)} \right| \leqq C_1 \cdot \|G\|_{L^2(B)} \cdot \varphi^t(U(t))^{1/2} + C_2 \cdot \varphi^t(U(t))$$

holds for every $t \in \mathcal{L}$ *and* $G \in \partial\varphi^t(U(t))$.

See also [6] and [9].

References

[1] Fujita, H. and N. Sauer: On existence of weak solutions of the Navier-Stokes equations in regions with moving boundaries. J. Fac. Sci. Univ. Tokyo, Sect. IA, **17**, 403–420 (1970).

[2] Inoue, A. and M. Wakimoto: On existence of solutions of the Navier-Stokes equation in a time dependent domain. ibid., **24**, 303–319 (1977).

[3] Ladyzhenskaya, O. A.: The Mathematical Theory of Viscous Incompressible Flow. Gordon and Breach, New York-London (1963).

[4] Lions, J. L.: Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires. Dunod, Paris (1969).

[5] Morimoto, H.: On existence of periodic weak solutions of the Navier-Stokes equations in regions with periodically moving boundaries. J. Fac. Sci. Univ. Tokyo, Sect. IA, **18**, 499–524 (1971-72).

[6] Ôtani, M. and Y. Yamada: On the Navier-Stokes equations in noncylindrical domains. An approach by the subdifferential operator theory. ibid., **25**, 185–204 (1978).

[7] Rabinowitz, P. H.: Existence and nonuniqueness of rectangular solutions of the Bénard problem. Arch. Rat. Mech. Anal., **29**, 32–57 (1968).

[8] Sattinger, D. H.: Group Theoretic Methods in Bifurcation Theory. Lecture Notes in Math., vol. 762, Springer (1978).

[9] Yamada, Y.: On evolution equations generated by subdifferential operators. J. Fac. Sci. Univ. Tokyo, Sect. IA, **23**, 495–515 (1976).