41. A Note on the Nash-Moser Implicit Function Theorem

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1. Problem and result. The purpose of this note is to give a simple and natural treatment of the Nash-Moser implicit function theorem established by Nash [2], Moser [1] and Schwartz [3]. (See also Sergeraert [4].) Nash's original idea (refined by Moser and Schwartz) seems to appear in the elegant form in our treatment.

Let $\{X_i\,;0\leq i<m\}$, $\{Y_i\,;0\leq i<m\}$ and $\{Z_i\,;0\leq i<m\}$, $11\leq m\leq\infty$, be three discrete Banach scales with norms $|\ \ |_i$, $\|\ \ \|_i$ and $\|\ \ \|_i$ such that $|\ \ |_i\leq|\ \ |_{i+1}$, $\|\ \ \|_i\leq\|\ \ \|_{i+1}$ and $\|\ \ \|_i\leq\|\ \ \|_{i+1}$. We assume $\{X_i\}$ is tame:

(X.0) There exists a (linear) smoothing operator $S(t)$, $t\in[1,\infty)$, from X_0 to X_{m-1} satisfying

(1) $|S(t)u|_k\leq c_0(k)t^{k-j}|u|_j,\quad u\in X_j,\quad j\leq k<m,$

(2) $|u-S(t)u|_j\leq c_0(k)t^{-(k-j)}|u|_k,\quad u\in X_k,\quad j\leq k<m.$

For a Banach space X with the norm $|\ \ |$, we put $X(r)=\{u\in X\,;|u|\leq r\}$. Let F be a continuous function from $X_1(R)\cap X_k\times Y_1(r)\cap Y_k$ into Z_{k-1} satisfying

(F.1) $\|F(u,y)\|_{k-1}\leq c_1(k)\{|u|_k+\|y\|_k\},\quad 1\leq k<m,$

(F.2) F has the Fréchet-derivative $F'_u(u,y)$ satisfying

(1) $F'_u(u,y)v$ is continuous from $X_1(R)\times Y_1(r)\times X_1$ into Z_0,

(2) $\|F'_u(u,y)v\|_0\leq c_2|v|_1$,

(3) $\|F(u+v,y)-F(u,y)-F'_u(u,y)v\|_0\leq c_2|v|_1^2$

for u and $v\in X_1(R)$ with $u+v\in X_1(R)$ and $y\in Y_1(r)$,

(F.3) $F'_u(u,y)$ has a right-inverse $L(u,y)$, $F'_u(\cdot)L(\cdot)=1$, satisfying

(1) $L(u,y)z$ is continuous from $X_1(R)\cap X_k\times Y_1(r)\cap Y_k\times Z_{k-1}$ into X_{k-1}, $1\leq k<m$,

(2) $|L(u,y)z|_0\leq c_3\|z\|_0,\quad u\in X_1(R),\quad y\in Y_1(r),\quad z\in Z_0,$

$|L(u,y)z|_{k-1}\leq c_3(k)\{(1+|u|_k+|y|_k)\|z\|_0+\|z\|_{k-1}\}$

for $u\in X_1(R)\cap X_k$, $y\in Y_1(r)\cap Y_k$ and $z\in Z_{k-1}$, $2\leq k<m$.

We consider the following equation

$$F(u,y)=0 \tag{1.1}$$

for each small $y\in Y_k$. We define a function $G(t,u,y)$ by

$$G(t,u,y)=u-S(t)L(u,y)F(u,y). \tag{1.2}$$

By Newton's method, we construct the approximate sequence $\{u_n\}$:

(1.3) $u_0=0,$

$u_{n+1}=G_{n+1}(y)=G(t_n,u_n,y)=u_n-S(t_n)L(u_n,y)F(u_n,y),$

$t_n=e^{\gamma\kappa^n},\quad \gamma\geq 1,\quad 4/3\leq\kappa\leq 3/2,\quad n=0,1,2,\cdots.$

With an appropriate choice of γ and k, u_n converges to a solution u of (1.1) in X_1 for each small y of $Y_1(r) \cap Y_k$. We have

Theorem (Nash-Moser). *Assume* (X.0), (F.1), (F.2) *and* (F.3). *Then, there exist integers* $k \geq 10$, $[(k+2)/3] \leq \ell < k-1$ *such that for any positive numbers* r_k *there exist positive numbers* r_1, R_ℓ *and a continuous function* $G(y)$ *from* $Y_1(r_1) \cap Y_k(r_k)$ *into* $X_1(R) \cap X_\ell(R_\ell)$ *satisfying*

$$F(G(y), y)=0 \quad \text{and} \quad G(0)=0. \tag{1.4}$$

2. Proof of Theorem. We write $c_i(1)=c_i$, $i=0,1,3$. We assume

$$|u_n|_1 \leq R, \quad u_n \in X_{m-1}, \quad 1 \leq n < n_1. \tag{2.1}$$

Then, we estimate $|h_{n+1}|_k$, $k \geq 10$, and $|h_{n+1}|_1$, where

$$\begin{aligned} h_{n+1} &= u_{n+1}-u_n = -S(t_n)L(u_n, y)F(u_n, y) \\ &= -S(t_n)L(u_n, y)\{F(u_n, y)-F(u_{n-1}, y)-F'_u(u_{n-1}, y)h_n\} \\ &\quad -S(t_n)L(u_n, y)F'_u(u_{n-1}, y)\{1-S(t_{n-1})\}L(u_{n-1}, y)F(u_{n-1}, y). \end{aligned} \tag{2.2}$$

We have for $u=u_j$, $0 \leq j < n_1$,

$$\|F(u, y)\|_{k-1} \leq c_1(k)(|u|_k+\|y\|_k), \tag{$2.3)_{k-1}$}$$

$$\|F(u, y)\|_0 \leq c_1(R+r)=c'_1, \tag{$2.3)_0$}$$

$$\begin{aligned} |L(u, y)F(u, y)|_{k-1} &\leq c_3(k)\{(1+|u|_k+\|y\|_k)\|F(u, y)\|_0+\|F(u, y)\|_{k-1}\} \\ &\leq c_3(k)c'_1(k)(1+|u|_k+\|y\|_k), \qquad c'_1(k)=c_1(k)+c'_1, \end{aligned} \tag{$2.4)_{k-1}$}$$

$$|L(u, y)F(u, y)|_0 \leq c_3c'_1. \tag{$2.4)_0$}$$

These inequalities imply

$$\begin{aligned} |h_{n+1}|_k &\leq t_n c_0(k)c_3(k)c'_1(k)(1+|u_n|_k+\|y\|_k) \\ &\leq t_n C(k)(1+|u_n|_k+\|y\|_k), \qquad C(k)=\max\{c_0(k)c_3(k)c'_1(k), 1\}. \end{aligned} \tag{2.5}$$

This implies

$$\begin{aligned} 1+|u_n|_k+\|y\|_k &\leq 1+\|y\|_k+\sum_{j=0}^{n-1}|h_{j+1}|_k \\ &\leq 1+\|y\|_k+\sum t_j C(k)\{1+|u_j|_k+\|y\|_k\}. \end{aligned} \tag{2.6}$$

Put

$$\beta_0=1+\|y\|_k \quad \text{and} \quad \beta_n=1+\|y\|_k+\sum_{j=0}^{n-1} t_j C(k)\beta_j, \qquad n \geq 1.$$

Then it follows that $\beta_n-\beta_{n-1}=t_{n-1}C(k)\beta_{n-1}$ and

$$\beta_n=\beta_0 \prod_{j=0}^{n-1}(1+t_jC(k)) \leq t_0 \cdots t_{n-1}(2C(k))^n(1+\|y\|_k), \tag{2.7}$$

$$1+|u_n|_k+\|y\|_k \leq \beta_n \leq t_0 \cdots t_{n-1}(2C(k))^n(1+\|y\|_k). \tag{2.8}$$

Using (F.2), (2.2), $(2.3)_0$ and $(2.4)_0$, we have

$$\begin{aligned} |h_{n+1}|_1 &\leq t_n c_0 c_3\|F(u_n, y)-F(u_{n-1}, y)-F'_u(u_{n-1}, y)h_n\|_0 \\ &\quad +t_n c_0 c_3 c_2|\{1-S(t_{n-1})\}L(u_{n-1}, y)F(u_{n-1}, y)|_1 \\ &\leq t_n c_0 c_3 c_2|h_n|_1^2+t_n c_0 c_3 c_2 t_{n-1}^{-(k-2)} c_0(k)c_3(k)c'_1(k)(1+|u_{n-1}|_k+\|y\|_k) \\ &\leq t_n C|h_n|_1^2+t_n t_{n-1}^{-(k-2)} CC(k)(1+|u_{n-1}|_k+\|y\|_k), \end{aligned} \tag{2.9}$$

where $C=c_0c_2c_3$. Substituting (2.8) into (2.9), we obtain

$$|h_{n+1}|_1 \leq t_n C|h_n|_1^2+t_n t_{n-1}^{-(k-2)} t_0 \cdots t_{n-2}(2C(k))^{n-1}CC(k)(1+\|y\|_k). \tag{2.10}$$

Now we put

$$|h_n|_1=e^{-\mu\gamma\kappa^n}s_n, \quad \mu \geq 2, \quad 1 \leq n \leq n_1. \tag{2.11}$$

Then we have

$$\begin{aligned} s_{n+1} &\leq e^{-\gamma\kappa^n a}Cs_n^2+e^{-\gamma\kappa^{n-1}b(k)}(2C(k))^{n-1}CC(k)(1+\|y\|_k), \\ a &= 2\mu-\mu\kappa-1=\mu(2-\kappa)-1 \geq 0, \\ b(k) &= k-2-\mu\kappa^2-\kappa-1/(\kappa-1)=k-\phi(\mu, \kappa). \end{aligned} \tag{2.12}$$

We choose k so that

(2.13) $$b(k)=k-\phi(\mu,\kappa)\geq 2b\in(0,1].$$

We have only to take a sufficiently large k. For example,

$$10-\phi(2,4/3)=1/9,\qquad 11-\phi(2,3/2)=1.$$

Taking the condition $a\geq 0$ into account, we obtain from (2.12)

(2.14) $$s_{n+1}\leq Cs_n^2+e^{-\gamma\kappa^{n-1}\cdot 2b+(n-1)d(k)}CC(k)(1+\|y\|_k),$$
$$d(k)=\log\{2C(k)\}\geq\log[2c_0(k)c_3(k)\{c_1(k)+c_1(R+r)\}].$$

Noting a trivial inequality

(2.15) $$\kappa^n\geq n\kappa',\qquad \kappa'=e\log\kappa\in(0.7,\ 1.1),$$

we determine $\gamma=\gamma(b,\varepsilon)$ by

(2.16) $$\gamma b=\max\{d(k)/\kappa',\ (d(k)-\log\varepsilon)/\kappa',\ 1\}.$$

Here $\varepsilon>0$ will be determined later. Then (2.14) reduces to

(2.17) $$s_{n+1}\leq Cs_n^2+C\varepsilon/2\ (1+\|y\|_k).$$

Assume

(2.18) $$D=1-2C^2\varepsilon(1+r_k)\geq 1/2,\qquad \|y\|_k\leq r_k,$$

and denote by $s>0$ the smaller root of the quadratic equation

$$s=Cs^2+C\varepsilon/2\ (1+r_k).$$

Then, there holds

$(2.19)_n$ $$s_n\leq s=C\varepsilon(1+r_k)/(1+D^{1/2})<2C\varepsilon(1+r_k)/3,\qquad 1\leq n\leq n_1,$$

if there holds

$(2.19)_1$ $$s_1=|u_1|_1\,e^{\mu\gamma\kappa}=|S(t_0)L(0,y)F(0,y)|_1\,e^{\mu\gamma\kappa}$$
$$\leq c_0c_3c_1\|y\|_1\,e^{\mu\gamma(1+\kappa)}=s'\leq s.$$

We determine ε here by

(2.20) $$\varepsilon=\min[1/\{4C^2(1+r_k)\},\ 9R/\{2C(1+r_k)\}].$$

Note that $(2.19)_1$ is a restriction on the size of $\|y\|_1$;

$(2.19)'_1$ $$\|y\|_1\leq r_1=\min\{r,\ se^{-(1+\mu)\gamma\kappa}/(c_0c_1c_3)\}.$$

If k, ε and γ satisfy (2.13), (2.20) and (2.16), respectively, and further $y\in Y_1(r_1)\cap Y_k(r_k)$, then we have

(2.21) $$|h_n|_1\leq e^{-\mu\gamma\kappa^n}s,\qquad 1\leq n\leq n_1,$$

(2.22) $$|u_n|_1\leq e^{-\mu\gamma\kappa'}s/\{1-e^{-\mu\gamma\kappa'}\}=s/(e^{\mu\gamma\kappa'}-1)$$
$$<s/(e^{2\kappa'}-1)<s/(e^{1.4}-1)<s/3\leq R.$$

Thus $\{u_n\}$ are well-defined. Moreover, (2.21) proves that $u_n=G_n(y)=G(t_n,u_{n-1},y)$ converges to a limit $u=G(y)$ in $X_1(R)$. Since $G_n(y)\in C^0(Y_1(r_1)\cap Y_k(r_k);X_1(R))$, $u=G(y)$ is also in the same space.

Using (F.2), (2.8) and (2.11), we can estimate $\|F(u_n,y)\|_0$:

(2.23) $$\|F(u_n,y)\|_0\leq\|F(u_n,y)-F(u_{n-1},y)-F'_u(u_{n-1},y)h_n\|_0$$
$$+\|F'_u(u_{n-1},y)\{1-S(t_{n-1})\}L(u_{n-1},y)F(u_{n-1},y)\|_0$$
$$\leq c_2|h_n|_1^2+c_2t_{n-1}^{-(k-2)}c_0(k)c_3(k)c'_1(k)(1+|u_{n-1}|_k+\|y\|_k)$$
$$\leq ce^{-2\mu\gamma\kappa^n}s+c'C(k)e^{-\gamma\kappa^{n-1}\{k-2-1/(\kappa-1)\}+(n-1)d(k)}.$$

By virtue of the following

(2.24) $$\gamma\{k-2-1/(\kappa-1)\}=\gamma\{k-\phi(\mu,\kappa)+\mu\kappa^2+\kappa\}$$
$$\geq\gamma\{2b+\mu\kappa^2+\kappa\}>2\gamma b>\gamma b+d(k)/\kappa',$$

$F(u_n,y)$ converges to 0 in Z_0. On the other hand $F(u_n,y)$ tends to $F(u,y)$, since F is continuous from $X_1(R)\times Y_1(r_1)$ to Z_0. This proves (1.4).

Since $\{X_i\}$ is tame, we can apply the interpolation inequality combined with (2.21) and (2.5)–(2.11), and obtain

$$|h_n|_\ell \leq c_0'(k)\,|h_n|_1^{(k-\ell)/(k-1)}\,|h_n|_k^{(\ell-1)/(k-1)} \tag{2.25}$$
$$\leq C'C(k)e^{-\gamma\kappa^n c(\ell)/(k-1)},$$
$$c(\ell)=\mu(k-\ell)-\{1/(\kappa-1)+b/\kappa\}\,(\ell-1).$$

Hence u_n converges in X_ℓ, if $c(\ell)>0$, which is equivalent to

$$\ell(1+\nu/\mu)<k+\nu/\mu, \qquad \nu=1/(\kappa-1)+b/k. \tag{2.26}$$

Note that $0<b\leq 1/2$, and hence $\nu<7/2<2\mu$ in our choice. Clearly $\ell=(k+2)/3$ satisfies (2.24). The proof is completed.

Remark 1. The solution $u\in X_1(R_1)\cap X(R_3)$ of (1.1) is unique, if there exists a left inverse $L(u, y)$ of $F'_u(u, y)$ satisfying (F.3).

Remark 2. If we assume $F(u, y)\in C^2(X_1(R)\cap X_j\times Y_1(r)\cap Y_j\,;\,Z_{j-1})$, more precisely, $F'(u, y)\in C^1(X_1(R)\cap X_j\times Y_1(r)\cap Y_j\,;\,B_s(X_{j-1}\times Y_{j-1}, Z_{j-1}))$, $j=1, 2$, then $G(y)\in C^2(Y_1(r_1)\cap Y_k\,;\,X_{\ell-2})$.

References

[1] J. Moser: Proc. Nat. Acad. Sci. USA, **47**, 1824–1831 (1961).
[2] J. Nash: Ann. of Math., **63**, 20–63 (1956).
[3] J. T. Schwartz: Nonlinear Functional Analysis. Gordon and Breach, New York-London-Paris (1969).
[4] F. Sergeraert: Ann. Sci. École Norm. Sup., **5**, 599–660 (1972).