37. Algebraic Equations for Green Kernel on a Tree

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Let Γ be a connected, locally finite tree with the set of vertices $V(\Gamma)$. Let A be a symmetric operator on $l^2(\Gamma)$, the space of square summable complex valued functions on $V(\Gamma)$:

(1) $$Au(\gamma)=\sum_{\langle\gamma',\gamma\rangle} a_{\gamma,\gamma'}u(\gamma')+a_{\gamma,\gamma}u(\gamma),$$

for $u\in l^2(\Gamma)$, with $a_{\gamma,\gamma}$ and $a_{\gamma,\gamma'}\in \boldsymbol{R}$ such that $a_{\gamma,\gamma'}\neq 0$, where $\langle\gamma,\gamma'\rangle$ means that γ and γ' are adjacent to each other. We assume that A is self-adjoint with the domain $\mathcal{D}(A)$: $\{u\in l^2(\Gamma)\,|\,\sum_{\gamma\in V(\Gamma)}|u(\gamma)|^2<\infty\}$. Then there exists uniquely the Green function $G(\gamma,\gamma'|z)$ for A, $\gamma,\gamma'\in V(\Gamma)$, representing the resolvent $(z-A)^{-1}$ for $z\in \boldsymbol{C}$, $\operatorname{Im} z\neq 0$:

(2) $$G(\gamma,\gamma'|z)=\int_{-\infty}^{+\infty}\frac{d\Theta(\gamma,\gamma'|\lambda)}{z-\lambda}$$

for the spectral kernel $\Theta(\gamma,\gamma'|\lambda)$ of A. We remark that for any $\gamma\in V(\Gamma)$, $G(\gamma,\gamma|z)$ satisfies

(3) $$\operatorname{Im} G(\gamma,\gamma|z)\cdot\operatorname{Im} z<0.$$

The purpose of this note is to extend a result obtained in [3] and [4] to an arbitrary tree. Algebraicity of Green functions was proved under various contexts. Here we want to give explicit formulae for them for an arbitrary self adjoint operator (see [3], [8] and [9]). First we want to prove

Lemma 1. *For arbitrary adjacent vertices γ,γ', suppose γ' and $\gamma_0\in V(\Gamma)$ do not lie in the same connected component of $\Gamma-\{\gamma\}$. Then the quotient $G(\gamma_0,\gamma'|z)/G(\gamma_0,\gamma|z)$ does not depend on γ_0.*

Proof. We denote by $\Gamma_{\gamma'}$ the connected subtree of Γ consisting of vertices γ'' lying in the connected component containing γ' of $\Gamma-\{\gamma\}$. We consider the following boundary value problem on the connected subtree $\Gamma_{\gamma'}\cup\{\gamma\}$ containing $\Gamma_{\gamma'}$ and γ: To find a solution $u\in l^2(\Gamma_{\gamma'}\cup\{\gamma\})$ such that

(4) $$Au(\gamma'')=zu(\gamma'') \qquad \text{for } \gamma''\in V(\Gamma_{\gamma'}),$$

(5) $$u(\gamma)=1.$$

Then every $G(\gamma_0,\gamma''|z)/G(\gamma_0,\gamma|z)$ is a solution for this problem. Hence Lemma 1 follows from the following:

Lemma 2. *There exists the unique solution $u(\gamma'')$ for the problem* (4) *and* (5).

Proof. Suppose that there exist two solutions $u_1(\gamma'')$ and $u_2(\gamma'')$ on $V(\Gamma_{\gamma'}\cup\{\gamma\})$. Then the difference $v=u_1-u_2$ also satisfies (4) and vanishes at γ. We have to prove that v vanishes identically. We define a function $\tilde{v}$ on $V(\Gamma)$ such that

(6) $$\tilde{v}(\gamma'')=v(\gamma'') \qquad \text{for } \gamma''\in V(\Gamma_{\gamma'}),$$

(7) $$\tilde{v}(\gamma'')=0 \qquad \text{otherwise.}$$

Then $\tilde{v}\in\mathcal{D}(A)$ and

(8) $$(z-A)\tilde{v}(\gamma'')=-\delta_{\gamma,\gamma''}a_{\gamma,\gamma'}v(\gamma').$$

Since the Green kernel $G(\omega,\omega'|z)$ for $\mathrm{Im}\,z\neq 0$ defines a bounded operator $\mathcal{G}(z)$ on $l^2(\Gamma)$ such that Image $\mathcal{G}(z)\subset\mathcal{D}(A)$ and

(9) $$1=(z-A)\cdot\mathcal{G}(z)\supset\mathcal{G}(z)\cdot(z-A),$$

we have for $\gamma''\in V(\Gamma)$

(10) $$\tilde{v}(\gamma'')=(z-A)\cdot\mathcal{G}(z)\tilde{v}(\gamma'')=\mathcal{G}(z)\cdot(z-A)\tilde{v}(\gamma'')=-a_{\gamma,\gamma'}G(\gamma'',\gamma|z)v(\gamma').$$

In particular for $\gamma''=\gamma$

(11) $$0=G(\gamma,\gamma|z)v(\gamma')$$

But $G(\gamma,\gamma|z)$ never vanishes, whence $v(\gamma')$ must vanish. Then (8) shows $\tilde{v}$ becomes an eigenfunction for A with the eigenvalue z. But A being self-adjoint, there is no non-trivial such function. Hence $\tilde{v}$, a fortiori, v must vanish identically. The lemma follows.

We shall denote the quotient $G(\gamma,\gamma'|z)/G(\gamma,\gamma|z)$ by $\alpha(\substack{\gamma\\ \gamma'}|z)$.

Now we want to prove a crucial

Lemma 3. *For adjacent vertices $\gamma,\gamma'\in V(\Gamma)$,*

(12) $$\alpha(\substack{\gamma\\ \gamma'}|z)=\frac{-W_\gamma+\sqrt{W_\gamma^2+4a_{\gamma,\gamma'}^2W_\gamma/W_{\gamma'}}}{2a_{\gamma,\gamma'}}.$$

Proof. By the equations for the Green functions, we have

(13) $$zG(\gamma,\gamma|z)-\sum_{\langle\gamma'',\gamma\rangle,\gamma''\neq\gamma'}a_{\gamma,\gamma''}G(\gamma,\gamma''|z)-a_{\gamma,\gamma'}G(\gamma,\gamma'|z)-a_{\gamma,\gamma}G(\gamma,\gamma|z)=1.$$

Dividing by $G(\gamma,\gamma|z)$,

(14) $$z-\sum_{\langle\gamma'',\gamma\rangle}a_{\gamma,\gamma''}\alpha(\substack{\gamma\\ \gamma''}|z)-a_{\gamma,\gamma}=W_\gamma(z).$$

In the same way

(15) $$zG(\gamma',\gamma|z)-\sum_{\langle\gamma'',\gamma\rangle,\gamma''\neq\gamma'}a_{\gamma,\gamma''}G(\gamma',\gamma''|z)-a_{\gamma,\gamma'}G(\gamma',\gamma'|z)-a_{\gamma,\gamma}G(\gamma',\gamma|z)=0.$$

Dividing by $G(\gamma',\gamma|z)$ we have

(16) $$z-\sum_{\langle\gamma'',\gamma\rangle,\gamma''\neq\gamma'}a_{\gamma,\gamma''}\alpha(\substack{\gamma\\ \gamma''}|z)-a_{\gamma,\gamma'}\frac{1}{\alpha(\substack{\gamma'\\ \gamma}|z)}-a_{\gamma,\gamma}=0.$$

Subtraction of (14) from (16) gives

(17) $$a_{\gamma,\gamma'}\left\{\frac{1}{\alpha(\substack{\gamma'\\ \gamma}|z)}-\alpha(\substack{\gamma\\ \gamma'}|z)\right\}=W_\gamma(z).$$

By symmetry we have similarly

(18) $$a_{\gamma,\gamma'}\left\{\frac{1}{\alpha(\substack{\gamma\\ \gamma'}|z)}-\alpha(\substack{\gamma'\\ \gamma}|z)\right\}=W_{\gamma'}(z).$$

These two equations give immediately the formulae (12), seeing the asymptotic behaviours of $W_\gamma(z)$, $W_{\gamma'}(z)$ and $\alpha(\substack{\gamma\\ \gamma'}|z)$:

(19) $$W_\gamma(z)\sim z,\ W_{\gamma'}(z)\sim z \quad\text{and}\quad \alpha(\substack{\gamma\\ \gamma'}|z)\sim\frac{a_{\gamma,\gamma'}}{z}.$$

The substitution of the formulae (12) into (14) yields the following equations:

Theorem. *For each vertex $\gamma\in V(\Gamma)$,*

(20) $$W_\gamma=z-a_{\gamma,\gamma}-\sum_{\langle\gamma,\gamma'\rangle}\frac{1}{2}\left(-W_\gamma+\sqrt{W_\gamma^2+4a_{\gamma,\gamma'}^2W_\gamma/W_{\gamma'}}\right).$$

For arbitrary $\gamma, \gamma' \in V(\Gamma)$, let $\gamma=\gamma_0, \gamma_1, \cdots, \gamma_{m-1}, \gamma_m=\gamma'$ be vertices on the geodesic path $[\gamma, \gamma']$ joining γ and γ' such that $\mathrm{dis}(\gamma, \gamma_j)=j$ where dis means the geodesic distance. Then

$$G(\gamma, \gamma' \mid z)=G(\gamma, \gamma \mid z) \prod_{j=1}^{m} \alpha\left(\begin{smallmatrix}\gamma_{j-1} \\ \gamma_j\end{smallmatrix} \mid z\right), \tag{21}$$

whence $G(\gamma, \gamma' \mid z)$ is completely determined by $W_\gamma(z)$ and (12).

Remark. The equations (20) are generally an infinite system of algebraic equations. They do not determine the holomorphic functions $W_\gamma(z)$ for $\mathrm{Im}\, z \neq 0$ even if we give the asymptotic behaviours by (19). But with the additional condition:

$$\mathrm{Im}\,\{a_{\gamma,\gamma'}\alpha(\begin{smallmatrix}\gamma \\ \gamma'\end{smallmatrix} \mid z)\}\cdot \mathrm{Im}\, z<0, \tag{22}$$

for $\mathrm{Im}\, z \neq 0$, $W_\gamma(z)$ are completely determined by (19) and (20). We shall discuss this problem elsewhere (see [5]).

References

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