4. *Index and Flow of Weights of Factors of Type III*

By Toshihiro HAMACHI and Hideki KOSAKI
Department of Mathematics, College of General Education,
Kyushu University

(Communicated by Kôsaku YOSIDA, M. J. A., Jan. 12, 1988)

§1. **Introduction.** V. Jones' theory on index of II_1 factors [5] is a major break-through in recent development of the theory of operator algebras. In the type II_1 case, if index is finite, then a factor and its subfactor are known to possess many similar properties (AFD, Property T, etc.). We would like to investigate a similar problem in the type III set-up.

Let $\mathcal{M}$ be a type III factor with a (type III) subfactor $\mathcal{N}$, and let E be a conditional expectation from $\mathcal{M}$ onto $\mathcal{N}$. The notion of index of E was introduced by the second-named author, [6], based on Connes' spatial theory and Haagerup's theory on operator valued weights, [4]. Throughout the article we assume Index $E<\infty$. To check how similar $\mathcal{M}$ and $\mathcal{N}$ are, we will compare the (smooth part of) flow of weights of $\mathcal{M}$ with that of $\mathcal{N}$. Our main theorem shows that each of the two flows is restricted by the other via the Index E ($<\infty$)-information. More precisely, there exists a single flow (X, T_t), and each of the two flows of weights appears as a (at most Index E to one) factor flow of (X, T_t).

In this announcement we will just sketch a proof of the main theorem. Full details and further results will be published elsewhere.

§2. **Notations and the main theorem.** Let E be a conditional expectation from a factor $\mathcal{M}$ onto its subfactor $\mathcal{N}$. We assume that Index E $<\infty$ and $\mathcal{M}$ and $\mathcal{N}$ are of type III. (If one of $\mathcal{M}$ and $\mathcal{N}$ is of type III, then the other is also of type III.) We will denote by $(X_{\mathcal{M}}, T_t^{\mathcal{M}})$ the flow of weights of $\mathcal{M}$ ([3]). The flow of weights can be computed from the associated crossed product $\tilde{\mathcal{M}}=\mathcal{M}\rtimes_\sigma \boldsymbol{R}$ and the dual action $\{\theta_s^{\mathcal{M}}\}_{s\in\boldsymbol{R}}$ on $\tilde{\mathcal{M}}$. (See [3], [10] for details.) More precisely, the center $Z(\tilde{\mathcal{M}})$ is isomorphic to $L^\infty(X_{\mathcal{M}}, d\mu)$, and by restriction the dual action induces the ergodic automorphism group on $Z(\tilde{\mathcal{M}})$. Then, the non-singular ergodic flow $\{T_t^{\mathcal{M}}\}_{t\in\boldsymbol{R}}$ on $X_{\mathcal{M}}$ is related to $\theta_t^{\mathcal{M}}$ via

$$(\theta_t^{\mathcal{M}}(f))(\omega)=f(T_{-t}^{\mathcal{M}}\omega)\,; \quad \omega\in X_{\mathcal{M}}, \quad t\in\boldsymbol{R}, \quad f\in Z(\tilde{\mathcal{M}})\cong L^\infty(X_{\mathcal{M}}, d\mu).$$

Theorem. *There exists a flow $(X, \{T_t\}_{t\in\boldsymbol{R}})$ satisfying the following:*

(i) *X is isomorphic to $X_{\mathcal{M}}\times\{1, 2, \cdots, m\}$ (resp. $X_{\mathcal{N}}\times\{1, 2, \cdots, n\}$) as a measure space for some positive integer $m, m\leq$ Index E (resp. positive integer $n, n\leq$ Index E),*

(ii) *the projection map $\pi_{\mathcal{M}}$ (resp. $\pi_{\mathcal{N}}$) from X onto $X_{\mathcal{M}}$ (resp. $X_{\mathcal{N}}$) intertwines T_t and $T_t^{\mathcal{M}}$ (resp. T_t and $T_t^{\mathcal{N}}$):*

$$T_t^{\mathcal{M}}\circ\pi_{\mathcal{M}}=\pi_{\mathcal{M}}\circ T_t, \quad T_t^{\mathcal{N}}\circ\pi_{\mathcal{N}}=\pi_{\mathcal{N}}\circ T_t, \quad t\in\boldsymbol{R}.$$

Let $\mathcal{M}$ and $\mathcal{N}$ be of type III_λ, III_μ, $0 \leqq \lambda, \mu \leqq 1$, respectively ([1]). In [7], it was shown that $\log \lambda / \log \mu$ is rational (when Index $E < \infty$). The above theorem gives us a bound for this rational number.

Corollary. *When* Index $E < \infty$, *we have*:

(i) $\lambda = 1$ *if and only if* $\mu = 1$,

(ii) $\lambda = 0$ *if and only if* $\mu = 0$,

(iii) *when* $0 < \lambda, \mu < 1$, *there exist two positive integers* p, q *such that* $p, q \leqq$ Index E *and* $\mu = \lambda^{p/q}$.

Existence of the common finite extension of two flows of weights shows that $T_t^{\mathcal{M}}$ is a trivial flow if and only if so is $T_t^{\mathcal{N}}$ and that $T_t^{\mathcal{M}}$ is periodic if and only if so is $T_t^{\mathcal{N}}$. Hence we get (i) and (ii). Furthermore, (i) in the previous theorem gives us a bound for the ratio between these two periods. Hence we get (iii).

§3. Sketch of a proof of the theorem. Representing $\mathcal{M}$ on $L^2(\mathcal{M})$, we construct the basic extension $\mathcal{M}_1 = \langle \mathcal{M}, e_{\mathcal{N}} \rangle$ and the conditional expectation $E_{\mathcal{M}} : \mathcal{M}_1 \to \mathcal{M}$ (from E^{-1}).

Let φ be a fixed normal faithful state on $\mathcal{N}$. Setting $\psi = \varphi \circ E \in \mathcal{M}_*^+$ and $\chi = \psi \circ E_{\mathcal{M}} \in (\mathcal{M}_1)_*^+$, we consider the inclusions

$$\tilde{\mathcal{M}}_1 = \mathcal{M}_1 \rtimes_{\sigma\chi} \boldsymbol{R} \supset \tilde{\mathcal{M}} = \mathcal{M} \rtimes_{\sigma\psi} \boldsymbol{R} \supset \tilde{\mathcal{N}} = \mathcal{N} \rtimes_{\sigma\varphi} \boldsymbol{R}$$

of von Neumann algebras of type II_∞ acting on $\mathcal{H} = L^2(\boldsymbol{R}, L^2(\mathcal{M}))$. Let $\mu(s)$, $s \in \boldsymbol{R}$, be the unitary operator on $\mathcal{H}$ defined by

$$(\mu(s)\xi)(t) = e^{-ist}\xi(t).$$

Then $\mathrm{Ad}\,\mu(s)$ gives rise to the dual actions $\theta_s^{\mathcal{M}_1}, \theta_s^{\mathcal{M}}$, and $\theta_s^{\mathcal{N}}$ on $\tilde{\mathcal{M}}_1, \tilde{\mathcal{M}}$, and $\tilde{\mathcal{N}}$ respectively.

We construct two flows from $\tilde{\mathcal{M}} \supset \tilde{\mathcal{N}}$. By Takesaki's criterion, [9], there exists a conditional expectation $\hat{E} : \tilde{\mathcal{M}} \to \tilde{\mathcal{N}}$ satisfying $\hat{\psi} \circ \hat{E} = \hat{\psi}$, where $\hat{\psi}$ is the dual weight on $\tilde{\mathcal{M}}$. As in [4], we have $\hat{E} \circ \theta_s^{\mathcal{M}} = \theta_s^{\mathcal{N}} \circ \hat{E}$, $s \in \boldsymbol{R}$. We then make use of the Pimsner-Popa inequality [8]:

$$E(x) \geqq (\text{Index } E)^{-1} x, \qquad x \in \tilde{\mathcal{M}}_+.$$

Actually we get the complete positivity of $x \to E(x) - (\text{Index})^{-1} x$, which shows that the Pimsner-Popa inequality remains valid for $\hat{E}$.

By restriction, $\hat{E}$ gives rise to

$$\hat{E} : \tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}' \longrightarrow Z(\tilde{\mathcal{N}}) \quad \text{and} \quad \hat{E} : Z(\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}') \longrightarrow Z(\tilde{\mathcal{N}}).$$

They still satisfy the Pimsner-Popa inequality and intertwine the dual actions on the respective algebras. A measure theoretical argument (or equivalently, the direct integral decomposition of $Z(\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}') \subset \tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}'$ ($\subset Z(\tilde{\mathcal{N}})'$) over $Z(\tilde{\mathcal{N}})$) shows that the spectrum of $Z(\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}')$ is of the form $X_{\mathcal{N}} \times \{1, 2, \cdots, n\}$, $n \leqq$ Index E. Here, dimension estimate obtained from the Pimsner-Popa inequality and ergodicity of the dual action on $Z(\tilde{\mathcal{N}})$ are crucial. The dual action on $Z(\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}') = L^\infty(X_{\mathcal{N}} \times \{1, 2, \cdots, n\})$ induces a non-singular (not necessarily ergodic) transformation T_t on $X_{\mathcal{N}} \times \{1, 2, \cdots, n\}$. Then T_t and $T_t^{\mathcal{N}}$ are intertwined by the projection map $\pi_{\mathcal{N}} : X_{\mathcal{N}} \times \{1, 2, \cdots, n\} \to X_{\mathcal{N}}$.

Repeating the same argument for $\tilde{\mathcal{M}}_1 \supset \tilde{\mathcal{M}}$, we conclude that

$$\begin{cases} Z(\tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}') \cong L^\infty(X_{\mathcal{M}} \times \{1, 2, \cdots, m\}),\ m \leqq \text{Index } E_{\mathcal{M}} = \text{Index } E, \text{ the flow} \\ T'_t \text{ on } X_{\mathcal{M}} \times \{1, 2, \cdots, m\} \text{ determined by the dual action on } Z(\tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}') \\ \text{and } T_t^{\mathcal{M}} \text{ on } X_{\mathcal{M}} \text{ are intertwined by the projection map } \pi_{\mathcal{M}} : X_{\mathcal{M}} \times \\ \{1, 2, \cdots, m\} \to X_{\mathcal{M}}. \end{cases}$$

It is easy to see that $\tilde{\mathcal{M}}_1$ is the basic extention of $\tilde{\mathcal{M}} \supset \tilde{\mathcal{N}}$, that is,

$$\tilde{\mathcal{M}}_1 = \tilde{J}\tilde{\mathcal{N}}'\tilde{J} = \langle \tilde{\mathcal{M}}, \hat{e} \rangle,$$

where $\hat{e}$ is the projection coming from $\hat{E}$ and $\tilde{J}$ is the modular conjugation operator on $\tilde{\mathcal{H}}$. Therefore, $\tilde{J} \circ {}^*\tilde{J}$ induces an anti-isomorphism between $\tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}'$ and $\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}'$, and hence an isomorphism between $Z(\tilde{\mathcal{M}}_1 \cap \tilde{\mathcal{M}}')$ and $Z(\tilde{\mathcal{M}} \cap \tilde{\mathcal{N}}')$. Since $\tilde{J}$ commutes with $\mu(s)$, this isomorphism actually intertwines the dual actions on the respective abelian algebras. Therefore, the two flows $(X = X_{\mathcal{N}} \times \{1, 2, \cdots, n\}, T_t)$ and $(X_{\mathcal{M}} \times \{1, 2, \cdots, m\}, T'_t)$ can be identified.

References

[1] A. Connes: Une classification des facteurs de type III. Ann. Sci. École Norm. Sup., **6**, 133–258 (1973).

[2] ——: Spatial theory of von Neumann algebras. J. Funct. Anal., **35**, 153–164 (1980).

[3] A. Connes and M. Takesaki: The flow of weights on factors of type III. Tohoku Math. J., **29**, 473–575 (1977).

[4] U. Haagerup: Operator valued weights in von Neumann algebras I. J. Funct. Anal., **32**, 175–306 (1979); II. ibid., **33**, 339–361 (1979).

[5] V. Jones: Index for subfactors. Invent. Math., **72**, 1–25 (1983).

[6] H. Kosaki: Extension of Jones' theory on index to arbitrary factors. J. Funct. Anal., **66**, 123–140 (1986).

[7] P. H. Loi: On the theory of index and type III factors. (Pennsylvania State Univ.) (1987) (preprint).

[8] M. Pimsner and S. Popa: Entropy and index for subfactors. Ann. Sci. École Norm. Sup., **19**, 57–106 (1986).

[9] M. Takesaki: Conditional expectation on von Neumann algebras. J. Funct. Anal., **9**, 306–321 (1972).

[10] ——: Duality for crossed products and the structure of von Neumann algebras of type III. Acta Math., **131**, 79–122 (1973).