32. On Some Inequalities in the Theory of Uniform Distribution. II

By Petko D. PROINOV and Nedyalka A. MITREVA
Department of Mathematics, University of Plovdiv, Bulgaria

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This is a continuation of Proinov and Mitreva [0].

2. In this section, we apply Theorem 1 to the theory of uniform distribution mod 1. Let $\sigma=(x_n)_1^\infty$ be a sequence of real numbers, and let g be a continuous distribution function on E. (A function g is called *distribution function* if it is nondecreasing on E with $g(0)=0$ and $g(1)=1$.) For an integer $N\geqq 1$ and $x\in E$, write $\Delta_N(\sigma;g;x)=A_N(\sigma;x)/N-g(x)$, where $A_N(\sigma;x)$ denotes the number of indices $n\leqq N$ such that the fractional parts $\{x_n\}$ are less than x. The sequence σ is called *asymptotically distributed* mod 1, with the asymptotic distribution function g, if $\lim_{N\to\infty}\Delta_N(\sigma;g;x)=0$ for all $x\in E$. The study of asymptotically distributed sequences was initiated by Schoenberg (see [10] or [2]).

Define the *discrepancies* $D_N(\sigma;g)$ and $D_N^*(\sigma;g)$ to be the oscillation and the supremum norm of $\Delta_N(\sigma;g;x)$, respectively. It is well known (see [4]) that both $\lim_{N\to\infty}D_N(\sigma;g)=0$ and $\lim_{N\to\infty}D_N^*(\sigma;g)=0$ are equivalent to the sequence σ being asymptotically distributed mod 1 with the distribution function g. In the next definition, we define the notion of φ-discrepancy which was given by Proinov [7] in the case $g(x)=x$.

Definition 2. Suppose that φ is a basic function, i.e., it is a nondecreasing positive function on $(0,\infty)$ with $\varphi(0+)=\varphi(0)=0$. Then for $N\geqq 1$, the *φ-discrepancy* $D_N^{(\varphi)}(\sigma;g)$ of σ with respect to the distribution function g, is defined by

$$D_N^{(\varphi)}(\sigma;g)=\int_0^1\varphi(|\Delta_N(\sigma;g;x)|)dx.$$

Theorem 2. *Let g be a continuous distribution function on E, and let φ be a basic function. Then the sequence σ is asymptotically distributed* mod 1 *with the distribution function g, if and only if*

$$\lim_{N\to\infty}D_N^{(\varphi)}(\sigma;g)=0.$$

This criterion in the case $\varphi(x)=x^p$ $(1\leqq p<\infty)$ was proved by Niederreiter [4], and in the case $g(x)=x$ by Proinov [7]. In the classical case $\varphi(x)=x^p$ and $g(x)=x$, Theorem 2 is due to Sobol' [11]. We omit the proof of Theorem 2 since it can be done in the same way as in the case $\varphi(x)=x^p$.

In the next theorem, we present two inequalities for the φ-discrepancy. They might be regarded as quantitative versions of Theorem 2 in the case where the distribution function g satisfies a Lipschitz condition on E.

Theorem 3. *Let φ be a basic function, and let g be a distribution function satisfying on E the Lipschitz condition with constant $L>0$. Then* *for the φ-discrepancy of the sequence σ with respect to the distribution function g, we have*

$$(1/L)F(D_N^*(\sigma\,;g)) \leqq D_N^{(\varphi)}(\sigma\,;g) \leqq \varphi(D_N^*(\sigma\,;g)) \tag{13}$$

and

$$(2/L)F\left(\frac{1}{2}D_N(\sigma\,;g)\right) \leqq D_N^{(\varphi)}(\sigma\,;g) \leqq \varphi(D_N(\sigma\,;g)), \tag{14}$$

where the function F is defined by (5).

In the case $g(x)=x$ these inequalities were obtained by Proinov ([7], [8]). In the classical case $\varphi(x)=x^p$ and $g(x)=x$, they are due to Niederreiter [6].

Proof. The upper bounds in (13) and (14) are trivial. The lower bounds are special cases of (3) and (4), respectively, since the function f defined on E by $f(x)=\Delta_N(\sigma\,;g\,;x)$ satisfies the requirements of Theorem 1. Q.E.D.

Remark. It is easy to see that Theorem 3 remains true also for discrepancies with respect to weights.

Reference*)

[0] P. D. Proinov and N. A. Mitreva: On some inequalities in the theory of uniform distribution. I. Proc. Japan Acad., 64A, 80–83 (1988).

*) Other references are given in [0].