14. On the Propagation of Analyticity for Some Class of Differential Equations with Non-involutive Double Characteristics

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1987)

1. Introduction. Let Ω be an open set in $\boldsymbol{R}^{n+1}$ containing the origin, with the coordinates $(x_0, \cdots, x_n)$. We shall consider the differential equation:

(1) $$P(x, D_x)u(x) = f(x), \quad f(x) \in \mathcal{A}(\Omega), \quad u(x) \in \mathcal{D}'(\Omega),$$

where $D_x = -i\partial/\partial x$, and $P(x, D_x)$ is a second order linear differential operator with analytic coefficients in Ω.

Let $p_2(x, \xi)$ be the principal symbol of $P(x, D_x)$. For k, l satisfying $k+l<n$ we put $(x', \xi') = (x_1, \cdots, x_k; \xi_1, \cdots, \xi_k)$, $(x'', \xi'') = (x_{k+1}, \cdots, x_{k+l}; \xi_{k+1}, \cdots, \xi_{k+l})$. We assume the following hypotheses:

(i) *p_2 has the form*

(2) $$p_2(x, \xi) = \xi_0^2 - a(x, \xi) + b(x, \xi),$$

where a, b are real valued and non-negative functions independent of ξ_0 and homogeneous of degree 2 with respect to ξ.

(ii) *$a(x, \xi)$ (resp. $b(x, \xi)$) vanishes exactly of order 2 on $\xi'=0$ (resp. $x''=\xi''=0$) in a conic neighborhood of $(0; 0, \cdots, 0, 1)$ in $T^*\Omega$.*

From (i), (ii) we can see that $p_2(x, \xi)$ has doubly characteristic points on $\Lambda = \{(x, \xi) \mid x'' = \xi_0 = \xi' = \xi'' = 0\}$ which is a non-involutive submanifold of $T^*\Omega$. We shall investigate the propagation of analyticity of a solution $u(x)$ of (1) along the leaf $\Gamma = \{(x, \xi) \mid x_i = 0,\ k+1 \leqq i \leqq n,\ \xi_i = 0,\ 0 \leqq i \leqq n-1,\ \xi_n = 1\}$ of Λ. We regard $(x_0, \cdots, x_k)$ as the coordinates of Γ and $(x_0, \cdots, x_k; \tilde{\xi}_0, \cdots, \tilde{\xi}_k)$ as those of $T^*\Gamma$. In order to state our theorem we introduce the function $q(x_0, x'; \tilde{\xi}_0, \tilde{\xi}')$ on $T^*\Gamma$ as follows:

(3) $$q(x_0, x'; \tilde{\xi}_0, \tilde{\xi}') = \tilde{\xi}_0^2 - \sum_{1 \leqq i, j \leqq k} \tilde{\xi}_i \tilde{\xi}_j \partial_{\xi_i} \partial_{\xi_j} a(x_0, x', 0; 0, \cdots, 0, 1)/2.$$

Let Σ_t be the subset of Γ defined as the intersection of the hypersurface $S_t = \{(x_0, x') \mid x_0 = t\}$ and the projection to Γ of the integral curves of

(4) $$H_q = 2\tilde{\xi}_0 \frac{\partial}{\partial x_0} - \frac{\partial q}{\partial \tilde{\xi}'} \frac{\partial}{\partial x'} + \frac{\partial q}{\partial x'} \frac{\partial}{\partial \tilde{\xi}'},$$

in $T^*\Gamma$ through a point $(0; \tilde{\xi}_0, \tilde{\xi}')$ such that $q(0; \tilde{\xi}_0, \tilde{\xi}') = 0$. Further let Ω_t be the connected component of $S_t \backslash \Sigma_t$ which is relatively compact. Then we have,

Theorem. *Let t_0, t_1 be positive real numbers such that $t_0 \geqq t_1$ and $\bigcup_{0 \leqq t \leqq t_0} \Omega_t \subset \Omega$, and assume that a solution $u(x)$ of (1) satisfies*

(5) $$WF_a(u) \cap \Sigma_{t_0} = \emptyset,$$

(6) $$WF_a(u)\cap\bigcup_{0\leqq t\leqq t_1}\Omega_t=\emptyset.$$

Then

(7) $$(0;0,\cdots,0,1)\notin WF_a(u).$$

Remark. We note that a result for equations with involutive double characteristics is obtained by Tose [6] by using results of [1], [2]. It gives a sharper result for the propagation of analyticity than ours, but it applies to (1) only when $b\equiv 0$.

2. Outline of the proof of Theorem. We introduce the Fourier-Bros-Iagolnitzer transformation of second type along Γ which is denoted by T_2. For $u(x)\in\mathcal{E}'(\boldsymbol{R}^{n+1})$, we define the function $T_2u(z,\lambda,\mu)$ on $\boldsymbol{C}^{n+1}\times\boldsymbol{R}^+\times(0,\mu_0]$ as follows:

$$T_2u(z,\lambda,\mu)=\int e^{-\lambda\mu^2(z^*-x^*)^2/2-\lambda(\mu z^{**}-x^{**})^2/2-i\lambda x_n}u(x)dx,$$

where $x^*=(x_0,\cdots,x_k)$, $x^{**}=(x_{k+1},\cdots,x_n)$ (see [3], [4]). This is holomorphic with respect to z. For $u(x)\in\mathcal{D}'(\Omega)$ we define the second microsupport of $u(x)$, which is a closed subset of $T^*\Gamma$ and denoted by $\mathrm{SS}^2_\Gamma(u)$, as follows: $(x_0^*,\xi_0^*)\notin\mathrm{SS}^2_\Gamma(u)$ if and only if there exists $\varepsilon>0$ such that

(8) $$|T_2(\chi u)|\leqq e^{\lambda\mu^2(|\mathrm{Im}\,z|^2/2-\varepsilon)},$$

on a neighborhood of $(x_0^*-i\xi_0^*,0)$ in $\boldsymbol{C}^{n+1}$ for $\lambda\geqq\lambda(\mu)$, where $\chi(x)\in C_0^\infty(\Omega)$, $\chi(x)\equiv 1$ near x_0 (see [4]). Let $\varphi(x^*)$ be a real valued analytic function on Γ such that $\varphi(x_0^*)=0$, $d\varphi(x_0^*)=\xi_0^*\neq 0$. According to [4] we have the following assertion called microlocal Holmgren theorem. If $WF_a(u)\cap\{(x,\xi)\,|\,(x,\xi)\in\Gamma,\ \varphi(x^*)<0\}=\emptyset$ and $(x_0^*,0;0,\cdots,0,1)\in WF_a(u)$, then $(x_0^*,\pm\xi_0^*)\in\mathrm{SS}^2_\Gamma(u)$. Therefore we can investigate the propagation of $WF_a(u)$ along Γ by estimating $\mathrm{SS}^2_\Gamma(u)$.

If we multiply both sides of (1) by $\chi(x)$ such that $\chi(x)\in C_0^\infty(\Omega)$ and $\chi(x)\equiv 1$ near 0, and operate T_2 to them, then the right-hand side satisfies (8) and the left-hand side can be rewritten as

$$T_2(\chi Pu)=\phi(P)T_2(\chi u),$$

where $\phi(P)$ is a classical pseudodifferential operator in the sense of Sjöstrand. Let

$$\phi(P)\sim\sum_{j=1}^{\infty}(\lambda\mu^2)^{-j}\tilde{p}_j(z,\zeta,\mu),$$

be its asymptotic expansion, then the principal symbol $\tilde{p}_0$ has the form as follows:

(9) $$\tilde{p}_0(z,\zeta,\mu)=\mu^2\tilde{b}(z,\zeta)+\mu^3r_3(z,\zeta)+\mu^4(\zeta_0^2-\tilde{a}(z,\zeta)+r_4(z,\zeta))+O(\mu^5),$$

where $\tilde{a}(z,\zeta)$ (resp. $\tilde{b}(z,\zeta)$) is the principal part of the Taylor expansion along $z^{**}=\zeta^*=\zeta^{**}=0$ of $a(z+i\zeta,\zeta+e_n)$ (resp. $b(z+i\zeta,\zeta+e_n)$) and $|r_3(z,\zeta)|\leqq C(|z^{**}|^2+|\zeta^{**}|^2)$, $|r_4(z,\zeta)|\leqq C(|z^{**}|^2+|\zeta^{**}|^2)$. We can see by means of (i), (ii) that

(10) $$\tilde{p}_0(z,\zeta,\mu)|_{\Lambda}\geqq c\mu^2(|z^{**}|^2+|\zeta^{**}|^2),$$

where $\Lambda=\{(z,\zeta)\,|\,\zeta=-\mathrm{Im}\,z\}$,

(11) $$\tilde{p}_0(z,\zeta,\mu)|_{\tilde{\Lambda}}\geqq\mu^4(\zeta_0^2-a(z,\zeta)),$$

where $\tilde{\Lambda}=\{(z,\zeta)\,|\,\zeta=-\mathrm{Im}\,z,\ z^{**}=\zeta^{**}=0,\ \zeta_0^2-a(z,\zeta)\geqq 0\}$.

(10), (11) imply that $\tilde{p}_0(z,\zeta,\mu)\neq 0$, i.e. $\phi(P)$ is elliptic when $(z,\zeta)\in\{(z,\zeta)\,|\,\zeta=-\mathrm{Im}\,z,\ z^{**}=\zeta^{**}=0,\ \zeta_0^2-a(z,\zeta)>0\}$ and μ is small enough. Then we can see that

(12) $$\mathrm{SS}_\Gamma^2(u)\subset\{(x^*,\tilde{\xi}^*)\,|\,q(x^*,\tilde{\xi}^*)\leqq 0\},$$

by an a priori estimate induced from ellipticity of $\phi(P)$.

Next we shall show the propagation of 2-microanalyticity of $u(x)$ along the integral curves of H_q from the cotangent spaces on Σ_{t_0}, which contain no points of $\mathrm{SS}_\Gamma^2(u)$, to $N=\{(x^*,\tilde{\xi}^*)\,|\,x^*=0,\ q(0,\tilde{\xi}^*)=0\}$. For a point $(x_0^*,\tilde{\xi}_0^*)$ satisfying that $q(x_0^*,\tilde{\xi}_0^*)=0$, and $(x_0^*,\tilde{\xi}_0^*)\notin\mathrm{SS}_\Gamma^2(u)$, there exists an open set U containing $(x_0^*,\tilde{\xi}_0^*)$ such that $\mathrm{SS}_\Gamma^2(u)\cap U=\varnothing$. Then we can find a real valued analytic function $\psi(x^*,\tilde{\xi}^*)$ which has the following properties:

(iii) The subset $\{(x^*,\tilde{\xi}^*)\,|\,\psi(x^*,\tilde{\xi}^*)<0\}\setminus U$ has a connected component which is relatively compact. We denote by V the component.

(iv) Let $\Phi(t,z)$ be a solution of the equation:

$$2\partial\Phi/\partial t-i\psi(z^*+2\partial\Phi/\partial z^*,\ -2i\partial\Phi/\partial z^*)=0,\qquad \Phi(0,z)=|\mathrm{Im}\,z|^2/2,$$

then Φ satisfies $\tilde{p}_0\,(z,-2i\partial\Phi/\partial z)\neq 0$ for $t>0$ small enough when $(\mathrm{Re}\,z^*,-\mathrm{Im}\,z^*+e_n)$ belongs to a neighborhood of $\overline{U\cup V}$.

(iii), (iv) imply that $V\cap\mathrm{SS}_\Gamma^2(u)=\varnothing$ by using an argument similar to the one in the microhyperbolic case by Sjöstrand (see [5]). Moreover choosing $\psi(x^*,\tilde{\xi}^*)$ such that the interior of V meets the integral curve of H_q through $(x_0^*,\tilde{\xi}_0^*)$, we can verify the propagation of 2-microanalyticity of $u(x)$ along the integral curves of H_q by the routine argument. Since every point in N can be connected by the integral curve with a point of the cotangent spaces on Σ_{t_0}, we obtain the assertion that

(13) $$\mathrm{SS}_\Gamma^2(u)\cap N=\varnothing.$$

Our theorem can be proved by using the method of sweeping out at the origin, because we can apply the microlocal Holmgren theorem to this case in view of (12), (13).

References

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