75. Euler Factors Attached to Unramified Principal Series Representations

By Takao WATANABE
Mathematical Institute, Tohoku University

(Communicated by Shokichi IYANAGA, M. J. A., Sept. 14, 1987)

1. Introduction. Let G be a connected reductive unramified algebraic group defined over a non-archimedean local field F of characteristic zero. We use the same notation as in [3]. We fix a non-degenerate character φ of $U(F)$. For a regular unramified character $\chi \in X_{\mathrm{reg}}(T)$ of $T(F)$, let $\rho(D_\chi)$ be the unique constituent of the unramified principal series representation $I(\chi)$ which has a Whittaker model with respect to φ (see [3] Theorem 2). The purpose of this note is to give a construction of an Euler factor attached to $\rho(D_\chi)$. A detailed account will be given elsewhere.

Since the minimal splitting field E of G is unramified over F, the Galois group Γ of E over F is cyclic. Let σ be a generator of Γ. Let $({}^LG^0, {}^LB^0, {}^LT^0)$ be a triple defined over the complex number field $\boldsymbol{C}$ which is dual to the triple (G, B, T). Let ${}^LG = {}^LG^0 \rtimes \Gamma$ be the finite Galois form of the L-group of G ([1]) and $X^*({}^LT^0)$ the character group of ${}^LT^0$. For $\gamma \in \Gamma$, $g \in {}^LG^0$ and $\lambda \in X^*({}^LT^0)$, the transform of g (resp. λ) by γ is denoted by ${}^\gamma g$ (resp. ${}^\gamma\lambda$). Let $\mathcal{R}({}^LG^0)$ (resp. $\mathcal{R}({}^LG)$) be the set of equivalence classes of finite dimensional irreducible representations of ${}^LG^0$ (resp. ${}^LG$).

2. The parametrization of $\mathcal{R}({}^LG)$. Let Λ be the set of dominant weights in $X^*({}^LT^0)$. Note that Λ is Γ-invariant. Let Λ/Γ be the set of Γ-orbits in Λ and $[\lambda]$ the Γ-orbit of $\lambda \in \Lambda$. For $[\lambda] \in \Lambda/\Gamma$, $e([\lambda])$ denotes the cardinality of $[\lambda]$. By the classical theory of Cartan and Weyl, there exists a bijection $R^\sim : \Lambda \to \mathcal{R}({}^LG^0)$ such that, for $\lambda \in \Lambda$, each representative of $R^\sim(\lambda)$ has the highest weight λ. For $\lambda \in \Lambda, \gamma \in \Gamma$ and a representative $R(\lambda)$ of $R^\sim(\lambda)$, we define the representation ${}^\gamma R(\lambda)$ of ${}^LG^0$ by ${}^\gamma R(\lambda)(g) = R(\lambda)({}^\gamma g)$, $g \in {}^LG^0$. Then ${}^\gamma R(\lambda)$ has the highest weight ${}^\gamma\lambda$. Thus we can take representatives $R(\lambda)$ of equivalence classes $R^\sim(\lambda)$ satisfying the following relation:

$$R({}^{\sigma^k}\lambda) = {}^{\sigma^k}R(\lambda) \quad \text{for any } \lambda \in \Lambda, \quad k = 0, 1, \cdots, e([\lambda]) - 1.$$

For $\lambda \in \Lambda$, the representation space of $R({}^\gamma\lambda)$, $\gamma \in \Gamma$ is denoted by $V_{[\lambda]}$. Hereafter, we fix a set of such representatives $\{(R(\lambda), V_{[\lambda]}) \mid \lambda \in \Lambda\}$.

We fix an orbit $[\lambda] \in \Lambda/\Gamma$ and put $e = e([\lambda])$. Let $\mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$ be the space of intertwining operators of $R(\lambda)$ into ${}^{\sigma^e}R(\lambda)$. This space is considered as a one dimensional subspace of $\mathrm{End}(V_{[\lambda]})$. Let $V_{[\lambda]}^\lambda$ be the common highest weight space of $R(\lambda)$ and ${}^{\sigma^e}R(\lambda)$. Then there exists a unique element $Q_{[\lambda]} \in \mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$ such that the restriction of $Q_{[\lambda]}$ to $V_{[\lambda]}^\lambda$ gives the identity map of $V_{[\lambda]}^\lambda$. Put $A_{[\lambda]} = \{\zeta_{|\Gamma|/e}^k \cdot Q_{[\lambda]} \mid k = 1, 2, \cdots, |\Gamma|/e\}$, where $\zeta_{|\Gamma|/e} = \exp(2\pi\sqrt{-1}e/|\Gamma|)$. Since one has $\mathrm{Hom}_{{}^LG^0}(R({}^\gamma\lambda), {}^{\sigma^e}R({}^\gamma\lambda)) = \mathrm{Hom}_{{}^LG^0}(R(\lambda), {}^{\sigma^e}R(\lambda))$

$=\boldsymbol{C}Q_{[\lambda]}$ as a subspace of End $(V_{[\lambda]})$ for every $\gamma \in \Gamma$, $A_{[\lambda]}$ depends only on the orbit $[\lambda]$. For $Q \in A_{[\lambda]}$, the representation $(R(\lambda, Q), V_{[\lambda]})$ of ${}^LG^0 \rtimes \langle \sigma^e \rangle$ is defined by $R(\lambda, Q)(g \rtimes \sigma^{ke}) = R(\lambda)(g) \cdot Q^k$, where $\langle \sigma^e \rangle$ is the cyclic group generated by σ^e. Further, let $r(\lambda, Q)$ be the representation of ${}^LG$ induced by $R(\lambda, Q)$. Then, it is shown that $r(\lambda, Q)$ depends only on the orbit $[\lambda]$, $Q \in A_{[\lambda]}$ and it is irreducible. Hence, if we denote by $r^{\sim}([\lambda], Q)$ the equivalence class containing $r(\lambda, Q)$, then we obtain a map $r^{\sim}: \coprod_{[\lambda] \in \Lambda/\Gamma} A_{[\lambda]} \to \mathcal{R}({}^LG)$, $([\lambda], Q) \mapsto r^{\sim}([\lambda], Q)$. By standard arguments in the representation theory we obtain the following

Theorem 1. *The map* $r^{\sim}: \coprod_{[\lambda] \in \Lambda/\Gamma} A_{[\lambda]} \to \mathcal{R}({}^LG)$ *is bijective.*

For $r = r^{\sim}([\lambda], \zeta^k_{|\Gamma|/e([\lambda])} \cdot Q_{[\lambda]}) \in \mathcal{R}({}^LG)$, we put

$$e(r) = e([\lambda]), \quad c(r) = 2\pi k(|\Gamma| \log(q_F))^{-1}\sqrt{-1} \quad \text{and} \quad \xi_r = \sum_{\lambda' \in [\lambda]} \lambda'.$$

Since the set of Γ-invariant elements in $X^*({}^LT^0)$ coincides with the cocharacter group $X_*(S)$ of S, ξ_r is contained in $X_*(S)$. Let $\mathcal{R}_0({}^LG)$ be the set of $r \in \mathcal{R}({}^LG)$ such that $\langle \alpha, \xi_r \rangle = 0$ for any $\alpha \in \Phi$, where $\langle\ ,\ \rangle: X^*(S) \times X_*(S) \to \boldsymbol{Z}$ is the natural pairing and Φ is the root system of G with respect to S. Further, let $\mathcal{R}_+({}^LG) = \mathcal{R}({}^LG) - \mathcal{R}_0({}^LG)$.

3. The construction of Euler factors. For $\chi \in X_{\mathrm{reg}}(T)$, let $\mathcal{WH}(\chi, \varphi)$ be the Whittaker model of $\rho(D_\chi)$ with respect to φ. For $r \in \mathcal{R}({}^LG)$, $f \in \mathcal{WH}(\chi, \varphi)$ and $s \in \boldsymbol{C}$, we define the zeta integral by

$$Z(s, r, f) = \int_{F^*} f(\xi_r(t))\, |t|_F^s \cdot \delta_B^{-1}(\xi_r(t))\, dt,$$

where dt is a Haar measure on the multiplicative group F^* of F. By a careful analysis of the behavior of Whittaker functions on $S(F)$, we obtain the following

Theorem 2. (1) *Let* $r \in \mathcal{R}_+({}^LG)$ *and* $\chi \in X_{\mathrm{reg}}(T)$. *Then, for any* $f \in \mathcal{WH}(\chi, \varphi)$, *the zeta integral* $Z(s, r, f)$ *is absolutely convergent for* $Re(s) \gg 0$.

(2) *For* $(r, \chi) \in \mathcal{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$, *let* $P(r, \chi)$ *be the set of polynomials* $P(X) \in \boldsymbol{C}[X]$ *such that* $P(q_F^{-s})Z(s, r, f)$ *is an entire function of* s *for any* $f \in \mathcal{WH}(\chi, \varphi)$. *Then* $P(r, \chi)$ *is a non-trivial principal ideal of* $\boldsymbol{C}[X]$ *and has the generator* $P_{r,\chi}(X)$ *with* $P_{r,\chi}(0) = 1$.

The generator $P_{r,\chi}(X)$ of $P(r, \chi)$ is uniquely determined by the pair (r, χ) and is independent of the choice of φ. The Euler factor attached to (r, χ) is defined to be $L(s, r, \chi) = P_{r,\chi}(q_F^{-s})^{-1}$. Another kind of Euler factor was defined by Langlands (see [1]). Denoting by $L(s, r, Sp(\chi))$ the Euler factor given by Langlands, we obtain

Theorem 3. *For any* $(r, \chi) \in \mathcal{R}_+({}^LG) \times X_{\mathrm{reg}}(T)$, $L(e(r)(s - c(r)), r, \chi)^{-1}$ *is a factor of* $L(s, r, Sp(\chi))^{-1}$ *as a polynomial of* q_F^{-s}.

Actually we can give a more explicit expression for $L(s, r, \chi)$ and comparing it with the corresponding part of $L(s, r, Sp(\chi))$ we get Theorem 3.

References

[1] A. Borel: Automorphic L-functions. Proc. Symposia Pure Math., **33**, part 2, 27–61 (1979).

[2] F. Rodier: Sur les facteurs eulériens associés aux sous-quotients des séries principales des groupes réductifs p-adiques. Journée Automorphes, Publication de l'Université Paris VII, vol. 15, pp. 107–133 (1982).

[3] T. Watanabe: The irreducible decomposition of the unramified principal series, Proc. Japan Acad., **63A**, 215–217 (1987).