66. *Harmonic Analysis on Negatively Curved Manifolds.* I

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(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 14, 1987)

Let M be a complete, simply connected, n-dimensional Riemannian manifold of the sectional curvature K_M satisfying $-b^2 \le K_M \le -a^2$ for some constants $a, b > 0$. The basic aim of this series of papers is to generalize the harmonic analysis on the open unit disc to the manifold M. In the first paper we treat Hardy spaces H^p, the space BMO and their probabilistic counterparts defined on the sphere at infinity $S(\infty)$ of M. Our research will deeply depends on recent remarkable works of M. T. Anderson, R. Schoen and D. Sullivan ([1], [2], [6]).

1. H^p and BMO. Throughout the paper we fix a point o in M. Let $\bar{M} = M \cup S(\infty)$. For $p \in M$ and $x \in \bar{M}$, we denote by $\gamma_{p,x}$ the uniquely determined unit speed geodesic ray with $\gamma_{p,x}(0) = p$ and $\gamma_{p,x}(t) = x$ for some $t \in (0, +\infty]$, and by $\dot{\gamma}_{p,x}(s)$ the tangent vector of $\gamma_{p,x}$ at s. Given $\delta > 0$, let $C(p, x, \delta)$ be the set $\{Q \in \bar{M} : \measuredangle_p(\dot{\gamma}_{p,x}(0), \dot{\gamma}_{p,Q}(0)) < \delta\}$, where $\measuredangle_p(v, w)$ is the angle between v and w in the tangent space at p. For simplicity, we put $Q(t) = \gamma_{0,Q}(t)$, $C(Q, t) = C(Q(t), Q, \pi/4)$ and $\Delta(Q, t) = C(Q, t) \cap S(\infty)$, for $Q \in S(\infty)$. We call $\Delta(Q, t)$ a surface ball.

Let Δ_M be the Laplace-Beltrami operator on M. A function f on M is harmonic in $D \subset M$, by definition, if $\Delta_M f(x) = 0$, $x \in D$. For $x \in M$, let $d\omega^x$ be the harmonic measure relative to x and M, and put $d\omega = d\omega^o$. If $f \in L^p$ $(= L^p(S(\infty), d\omega))$ $(1 \le p \le \infty)$, then we denote by $\tilde{f}$ the harmonic extension of f, i.e. $\tilde{f}(x) = \int f(Q)\, d\omega^x(Q)$ when $x \in M$, and $\tilde{f}(x) = f(x)$ when $x \in S(\infty)$. Let $N(f)$ be the nontangential maximal function of f, that is, $N(f)(Q) = \sup\{|\tilde{f}(z)| : z \in \Gamma(Q)\}$, $Q \in S(\infty)$, where $\Gamma(Q) = \{x \in M : Q \in C(x, \gamma_{o,x}(+\infty), \pi/4) \cap S(\infty)\}$. The set $\Gamma(Q)$ is an analogue of Stoltz domains. Hardy spaces on $S(\infty)$ are defined by $H^p = \{f \in L^1 : \|f\|_{H^p} = \|N(f)\|_p < +\infty\}$, $0 < p \le \infty$, where $\|g\|_p = (\int |g|^p d\omega)^{1/p}$, for every measurable function g on $S(\infty)$.

From a modification of the proof of [2, Theorem 7.3], it follows that $H^p = L^p$, $1 < p \le \infty$, but, in general, H^1 is a proper subspace of L^1. C. Fefferman's duality theorem asserts that the dual space of H^1 on $\boldsymbol{R}^n$ is the space BMO. In our context, the space BMO is defined as follows: For $f \in L^1$, let $\|f\|_* = \sup\left\{\frac{1}{\omega(\Delta)}\int_\Delta \left| f(q) - \frac{1}{\omega(\Delta)}\int_\Delta f\, d\omega \right| d\omega(q) : \Delta \text{ is a surface ball}\right\}$ and BMO $= \{f \in L^1 : \|f\|_* < +\infty\}$.

One of our main results is the following:

Theorem 1. (1) *For every* $f\in L^2$ *and* $g\in \mathrm{BMO}$,

$$\left|\int fgd\omega\right|\leq C\|f\|_{H^1}\|g\|_*,$$

where C is a constant depending only on n and curvature bounds.

(2) *Suppose M satisfies the following condition* (C):

$$\text{(C)}\ \begin{cases}\text{For every } Q\in S(\infty),\ t>0 \text{ and } z\in C(Q,t),\\ \Delta(\gamma_{o,z}(+\infty),t)\cap\Delta(Q,t)\neq\emptyset.\end{cases}$$

Then for every continuous linear functional F on H^1, *there exists an element g of* BMO *such that* $F(f)=\int fgd\omega$ $(f\in L^2)$.

Moreover, the norm of F is equivalent to $\|g\|_*$.

Remark. It is easy to see that if M is rotationally symmetric at o, then M satisfies the condition (C).

To prove Theorem 1 we use probabilistic Hardy spaces which will be studied in the next section.

2. Probabilistic version of H^p and BMO. We will investigate probabilistic H^p and BMO. For notations and terminologies of probability theory concepts, the reader is referred to Ikeda and Watanabe [4] and Sullivan [6]. Let $\{P_x\}_{x\in M}$ be a Δ_M-diffusion and put $\mathcal{F}_t=\mathcal{F}_t(\overline{W}(M))$, $t\in[0,\infty]$. We set $P=P_0$, $E[\ ]=E_0[\ ]$ and $E[\ |\mathcal{F}_t]=E_0[\ |\mathcal{F}_t]$, $t\geq 0$. For $w\in\overline{W}(M)$ and $f\in L^1$, let $X_t(w)=w(t)$ and $Mf_t=\tilde{f}(X_t)$, $t\geq 0$. Probabilistic versions of H^p, $0<p\leq\infty$, and BMO are defind by

$$H^p_{pr}=\{f\in L^1:\|f\|_{p,pr}=(E[\sup_t|Mf_t|^p]^{1/p}<+\infty\}$$

and

$$\mathrm{BMO}_{pr}=\{f\in L^1:\|f\|_{*,pr}=\sup_t\|E[|Mf_\infty-Mf_t||\mathcal{F}_t]\|_\infty<+\infty\}.$$

Theorem 2. (1) $H^p\subset H^p_{pr}$, $0<p\leq\infty$, *and* $\mathrm{BMO}\subset\mathrm{BMO}_{pr}$. *Moreover,* $\|f\|_{p,pr}\leq C_p\|f\|_{H^p}$ $(0<p\leq\infty)$ *and* $\|f\|_{*,pr}\leq c\|f\|_*$, *where* C_p *is a constant depending only on n, a, b and p, and c on n, a and b.*

(2) *If M satisfies the condition* (C), *then* $H^p=H^p_{pr}$, $1\leq p\leq\infty$, *and* $\mathrm{BMO}=\mathrm{BMO}_{pr}$. *Norms* $\|\ \|_{H^p}$ *and* $\|\ \|_*$ *are equivalent to* $\|\ \|_{p,pr}$ *and* $\|\ \|_{*,pr}$ *respectively.*

3. Spaces of homogeneous type. In order to prove Theorems 1 and 2, we give a variant of the notion of space of homogeneous type introduced by Coifman and Weiss [3]. Our method does not depend on metric and is adapted to the study of analysis on $S(\infty)$. Let W be a topological space and μ be a positive Borel measure on W. Suppose for every $Q\in S(\infty)$ and $t\in\boldsymbol{R}$, there exists an open neighborhood $\Delta_t(Q)$ (called surface ball) of Q satisfying

(1) $W=\lim_{t\to-\infty}\Delta_t(Q)\supset\Delta_r(Q)\supset\supset\Delta_{r+s}(Q)\supset\lim_{t\to+\infty}\Delta_t(Q)=\{Q\}$, $Q\in W$, $r\in\boldsymbol{R}$, $s>0$, where $A\supset\supset B$ means that A contains the closure of B;

(2) there is a constant $k>0$ such that for every Q, $R\in W$ and $r\in\boldsymbol{R}$, $\Delta_r(Q)\cap\Delta_r(R)\neq\emptyset$ implies $\Delta_{r-k}(Q)\supset\Delta_r(R)$;

(3) $0<\mu(\Delta_r(Q))<+\infty$ for every $\Delta_r(Q)$;

(4) there exists a constant C such that $\mu(\Delta_{r-1}(Q))\leq C\mu(\Delta_r(Q))$ for every $\Delta_r(Q)$.

It is easy to check that if (W, μ) is a space of homogeneous type with a quasi-metric ρ, then balls $\{x \in W : \rho(x, Q) < e^{-r}\}$, $Q \in W$, $r \in \boldsymbol{R}$, satisfy conditions (1)—(4). In addition to this, by Anderson and Schoen [2], the conditions (1)—(4) are satisfied by surface balls in $S(\infty)$ and every harmonic measure. They also pointed out that Vitali type covering lemma holds good on $S(\infty)$. We note that the covering lemmas of Vitali and Whitney type can be proved in the more general space $(W, \mu; \{\Delta_t(Q)\})$. These facts imply that the atomic Hardy space theory introduced by Coifman and Weiss [3] is carried out in our setting.

Let $0<p\leq 1$. A function $a \in L^1(\mu)$ is a p-atom if $\int a d\mu = 0$ and there is a surface ball Δ containing the support of a, with $\|a\|_\infty^p \leq 1/\mu(\Delta)$. Let $H^1_{at} = \{h \in L^1 : \|h\|_{1,at} = \inf\{\sum_{j=0}^\infty |\lambda_j| : h = \sum_{j=0}^\infty \lambda_j a_j,\ \lambda_j \in \boldsymbol{R}$ and a_j is 1-atom, $j=0, 1, \cdots\} < +\infty\}$. Here we regard $\inf \varnothing$ as $+\infty$ and $\mu(W)^{-1}$ as a 1-atom if $\mu(W) < \infty$. Define BMO on W by the same way as BMO on $S(\infty)$.

Theorem 3. *The dual space of* H^1_{at} *is identified with* BMO.

If $(W, \mu) = (S(\infty), \omega)$, we obtain the following:

Theorem 4. *There exists a constant C depending only on n, a and b such that $\|f\|_{1,at} \leq C\|f\|_{H^1}$ for every $f \in L^2$. Moreover, if M satisfies the condition* (C), *then $H^1 = H^1_{at}$ and norms $\|\ \|_{1,at}$ and $\|\ \|_{H^1}$ are equivalent.*

4. Sketch of proofs of Theorems 1, 2 and 4. We will use $C_1, C_2, \cdots$ to denote positive constants depending only on n, a and b. A difficult part of Theorem 1 is proved by applying a result in martingale theory:

$$\left|\int f g d\omega\right| = |E[Mf_\infty Mg_\infty]| \leq C\|Mf\|_{1,pr}\|g\|_{*,pr} \leq C_1\|f\|_{H^1}\|g\|_*,$$

(by martingale theory)

where C is a universal constant. In order to prove the remainder part we will show the following estimate of the Poisson kernel $K(\ ,\ )$:

Lemma. *There exists $r_0 > 0$ satisfying the following: Let N be a positive integer and r be a number larger than $Nr_0 + 1$. Let $\{m_j\}_{j=1,\dots,k}$ $(k \leq N)$ be any sequence such that*

(1) $0 = m_1 < \cdots < m_k < r-1$;

(2) $m_{j+1} - m_j \geq r_0$ $(j = 0, \cdots, k-1)$.

Then $|[K(x, Q_1)/K(x, Q_0)] - 1| \leq C_3 2^{-j}$, for every $Q_0 \in S(\infty)$ and $Q_1 \in \Delta(Q_0, r)$ and $x \in M - C(Q_0(r - m_j), Q_0, \pi/4)$.

Note that the estimate of Lemma is originally due to Jerison and Kenig [5] in the case of Euclidean domains. Now, we assume the condition (C). Then from this lemma and the condition (C) we can prove that $\|A\|_{H^1} \leq C_2$ for every atom A. Hence we obtain that $H^1_{at} \subset H^1$ under the condition (C). Remembering Theorem 3 and inequalities obtained in this section we can prove Theorems 1 and 4 by using functional analysis. Further, Theorem 1 and a duality argument imply that $H^1 \supset H^1_{pr}$ and $\mathrm{BMO} \supset \mathrm{BMO}_{pr}$, from which Theorem 2 follows.

A detailed proof will be published elsewhere.

References

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