65. On Representations of Lie Superalgebras

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In this note we give a method of constructing irreducible (unitary) representations of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, using (unitary) representations of a usual Lie algebra $\mathfrak{g}_0$, the even part of $\mathfrak{g}$. We propose some problems about this method of extension. If we can solve these problems, the classification of irreducible representations will be achieved. We analyze this method of extension and give some examples.

1. Unitary representations. Let (π, V) be an irreducible representation of $\mathfrak{g}$ on a $\mathbf{Z}_2$-graded complex vector space $V = V_0 + V_1$ in the sense of Kac [4, § 1]. On the even part V_0 and also on the odd part V_1 of V, we have naturally representations of the even part $\mathfrak{g}_0$, of which π is called an extension. We call π *unitary* if V is equipped with a positive definite inner product $\langle \cdot, \cdot \rangle$ in V satisfying

(i) $V_0 \perp V_1$ (orthogonal) under $\langle \cdot, \cdot \rangle$, and

(ii) $\langle \cdot, \cdot \rangle$ is $\mathfrak{g}$-invariant in the sense that

$$\langle i\pi(X)v, v' \rangle = \langle v, i\pi(X)v' \rangle \qquad (v, v' \in V, \ X \in \mathfrak{g}_0),$$
$$\langle j\pi(\xi)v, v' \rangle = \langle v, j\pi(\xi)v' \rangle \qquad (v, v' \in V, \ \xi \in \mathfrak{g}_1),$$

where $i = \sqrt{-1}$ and j is a fixed forth root (depending only on π) of -1, i.e., $j^2 = \varepsilon i$ with $\varepsilon = \pm 1$. We call j^2 the associated constant for π since the essential thing is not j itself but $j^2 = \varepsilon i$. In this case, both $\pi(\mathfrak{g}_0) | V_0$ and $\pi(\mathfrak{g}_0) | V_1$ are usual unitary representations of $\mathfrak{g}_0$.

2. Extension problems. To classify and to construct all the irreducible (unitary) representations of $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, we wish to utilize rich results on representations of usual Lie algebra $\mathfrak{g}_0$. Therefore we propose some problems from this point of view.

Problem 1. Take an *irreducible* representation ρ of $\mathfrak{g}_0$ on a complex vector space V_0. Then, do there exist any irreducible representations (π, V) of $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ extending (ρ, V_0)? (More exactly, V_0 is imbedded into V as its subspace of degree 0, and ρ is equivalent to $\pi(\mathfrak{g}_0) | V_0$ under this embedding.) If they do exist, construct all of them.

Problem 2. Let (ρ, V_0) be an *irreducible unitary* $\mathfrak{g}_0$-module. Then do there exist any irreducible unitary extensions of (ρ, V_0) to $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$? If any, in which different ways can we extend it?

After some study, we recognize that the even part (ρ, V_0) of an irreducible representation (π, V) is not irreducible in many cases. As a matter of fact, the adjoint representation of a Lie superalgebra of type A is in such a case. So we generalize the above problems to Problem 1 bis (or

Problem 2 bis), where we start from (ρ, V_0) of g_0 not necessarily irreducible.

3. **Equations for extension.** For a representation (π, V) of g, we define a bilinear map $B : g_1 \times g_1 \to gl(V_0)$ by

(1) $$B(\xi, \eta) = \pi(\xi)\pi(\eta) \mid V_0.$$

If π is irreducible, we can reconstruct (π, V) using (ρ, V_0) and B as follows: let $g_{1,c}$ be the complexification of g_1, considered as g_0-module. Put $W = g_{1,c} \otimes_c V_0$, the tensor product as g_0-modules, and let p be the canonical g_0-homomorphism of W into V_1 given by

$$p : W \ni \xi \otimes v \longmapsto \pi(\xi)v \in V_1 \qquad (\xi \in g_1, \ v \in V_0).$$

Then p is surjective. Let m be the kernel of p, and put $\tilde{W} = W/m$ and denote by $[w]$ the element in $\tilde{W}$ represented by $w \in W$. Then $\tilde{W} \cong V_1$ as g_0-modules through p.

Theorem 1. *Let* (ρ, V_0) *be a representation of the even part* g_0 *of* g, *not necessarily irreducible, and* (π, V), $V = V_0 + V_1$, *be an extension of* (ρ, V_0), *satisfying:* (PRO 1) $\pi(g_1)V_0 = V_1$; *and* (PRO 2) *an element* $v_1 \in V_1$ *is equal to* 0 *if and only if* $\pi(\xi)v_1 = 0$ *for any* $\xi \in g_1$. *Put* $B(\xi, \eta)$, $\xi, \eta \in g_1$, *as in* (1). *Then* B *satisfies the following system of equations:*

(EXT 1) $B(^x\xi, \eta) + B(\xi, {}^x\eta) = [\rho(X), B(\xi, \eta)]$,

(EXT 2) $B(\xi, \eta) + B(\eta, \xi) = \rho([\xi, \eta])$,

(EXT 3) $B(\tau, \xi)B(\eta, \zeta) + B(\tau, \eta)B(\xi, \zeta) = B(\tau, {}^{[\xi, \eta]}\zeta) + B(\tau, \zeta)\rho([\xi, \eta])$,

for $\tau, \xi, \eta, \zeta \in g_1$ *and* $X \in g_0$, *where* ${}^x\xi = [X, \xi]$.

Theorem 2. *Let* (ρ, V_0) *be as above. Assume that we are given a bilinear map* B *from* $g_1 \times g_1$ *into* $gl(V_0)$, *which satisfies* (EXT 1)–(EXT 3). *Put* $W = g_{1,c} \otimes_c V_0$ *and define its* g_0-*submodule* m *by*

$$m = \{\textstyle\sum_i \eta_i \otimes v^i \; ; \; \eta_i \in g_1, \ v^i \in V_0, \ \textstyle\sum_i B(\xi, \eta_i)v^i = 0 \text{ for all } \xi \in g_1\}.$$

Take $\tilde{W} = W/m$ *as the space* V_1 *of degree* 1, *and define* g_1-*action on* $V = V_0 + V_1$ *by*

$$\pi(\xi) : \tilde{W} \ni [\eta \otimes v] \longmapsto B(\xi, \eta)v \in V_0,$$

$$\pi(\xi) : V_0 \ni v \longmapsto [\xi \otimes v] \in \tilde{W}, \qquad \text{for } \xi, \eta \in g_1, \ v \in V_0.$$

Then (π, V) *is an extension of* (ρ, V_0) *satisfying* (PRO 1), (PRO 2). *Moreover* π *is irreducible if and only if it has the property:*

(PRO 3) *The subalgebra* $\langle \rho(g_0), B(g_1, g_1) \rangle$ *of* $gl(V_0)$, *generated by* $\rho(g_0)$ *and* $B(g_1, g_1)$, *acts on* V_0 *irreducibly.*

Any irreducible extension can be obtained in this way up to equivalence.

Theorem 3. *Let* (ρ, V_0) *be a unitary representation of* g_0, *and* (π, V), $V = V_0 + V_1$, *be its extension with* (PRO 1), (PRO 2), *which is given canonically by* B *satisfying* (EXT 1)–(EXT 3). *Then* (π, V) *can be made unitary if and only if there exists a* g_0-*invariant positive definite inner product* $\langle \cdot, \cdot \rangle_0$ *on* V_0 *for which the following condition holds:*

(UNI) $$j^2 \sum_{k, m} \langle B(\xi_m, \xi_k)v^k, v^m \rangle_0 \geq 0 \qquad (\xi_k \in g_1, \ v^k \in V_0).$$

When we apply (EXT 1)–(EXT 3) to certain types of simple Lie super-algebras, it is more convenient to use, instead of B, a skew-symmetric bilinear map $A : g_1 \times g_1 \to gl(V_0)$,

$$A(\xi,\eta)=B(\xi,\eta)-B(\eta,\xi) \qquad \text{for } \xi,\eta \in \mathfrak{g}_1.$$

We reduce the system of equations (EXT 1)–(EXT 3) to more simpler one. (EXT 2) is dissolved into the skew-symmetricity of A. We define a linear map P_B from $\mathfrak{g}_1^{(4)}=\mathfrak{g}_{1,c}\otimes\mathfrak{g}_{1,c}\otimes\mathfrak{g}_{1,c}\otimes\mathfrak{g}_{1,c}$ to $\mathfrak{gl}(V_0)$ as follows: for $\tau\otimes\xi\otimes\eta\otimes\zeta$ with $\tau,\xi,\eta,\zeta \in \mathfrak{g}_{1,c}$,

$$P_B(\tau\otimes\xi\otimes\eta\otimes\zeta)=B(\tau,\xi)B(\eta,\zeta)+B(\tau,\eta)B(\xi,\zeta)-B(\tau,{}^{[\xi,\eta]}\zeta)-B(\tau,\zeta)\rho([\xi,\eta]).$$

Denote by S_{pq} the automorphism of $\mathfrak{g}_1^{(4)}$ exchanging the p-th and q-th factors in decomposable vectors. Then, taking into account the $\mathfrak{g}_0$-equivariancy of the maps P_B and A, we get the following

Theorem 4. *Let* $\{u_j \in \mathfrak{g}_1^{(4)}; 1\leq j\leq M\}$ *be a system of generators of* $\mathfrak{g}_1^{(4)}$ *as* $\langle U(\mathfrak{g}_{0,c}),S_{23}\rangle$*-module, and* $\{z_k \in \mathfrak{g}_{1,c}\wedge\mathfrak{g}_{1,c}; 1\leq k\leq N\}$ *be that of the exterior product* $\mathfrak{g}_{1,c}\wedge\mathfrak{g}_{1,c}$ *of* $\mathfrak{g}_{1,c}$ *as* $\mathfrak{g}_0$*-module.*

(i) *Under the condition* (EXT 1), *the equation* (EXT 3) *on B is equivalent to*

(EXT 3*) $\qquad\qquad P_B(u_j)=0 \qquad (1\leq j\leq M).$

(ii) *Put* $A(\xi\wedge\eta)=A(\xi,\eta)$ *for* $\xi,\eta \in \mathfrak{g}_1$ *and* $A_k=A(z_k) \in \mathfrak{gl}(V_0)$, $1\leq k\leq N$. *Then the condition* (EXT 1) *on B is equivalent to the following on* $\{A_1,A_2,\cdots,A_N\}$:

(EXT 1*) $\qquad$ *if* $\sum_{1\leq k\leq N} {}^{x_k}z_k=0$ *with* $x_k \in U(\mathfrak{g}_{0,c})$, *then necessarily*
$$\sum_{1\leq k\leq N} {}^{x_k}A_k=0,$$

where the action of $x_k \in U(\mathfrak{g}_{0,c})$ *on* $A_k \in \mathfrak{gl}(V_0)$ *is canonically induced from that of* $X \in \mathfrak{g}_0: \mathfrak{gl}(V_0)\ni C\mapsto[\rho(X),C]\in\mathfrak{gl}(V_0).$

(iii) *The system of equations* (EXT 1)–(EXT 3) *on B is equivalent to that of equations* (EXT 1*), (EXT 3*) *under the skew-symmetricity of A.*

4. Some examples. Let $\mathfrak{g}=\mathfrak{osp}(2/1)$ be a real form of $\mathfrak{osp}(1,2)$ of type $B(0,1)$, with $\mathfrak{g}_0\cong\mathfrak{sp}(2;\boldsymbol{R})$. Then, for any irreducible representation (π,V) of $\mathfrak{g}$, both $\pi(\mathfrak{g}_0)|V_0$ and $\pi(\mathfrak{g}_0)|V_1$ are irreducible. Moreover we can solve Problems 1 bis and 2 bis, and thus get the classification and the realization of all the irreducible (unitary) representations of $\mathfrak{g}$. For $\mathfrak{osp}(2n/1)$, cf. [1] and [3].

Next we consider Problem 2 (or 2 bis) for Lie superalgebras of type A. For $\mathfrak{g}=\mathfrak{sl}(m,n;K)$ or $\mathfrak{l}(m,n;K)$ ($K=\boldsymbol{R}$ or $\boldsymbol{C}$), it has only one irreducible unitary representation, the trivial one. Now take one of real forms of $A(1,0)\cong\mathfrak{sl}(2,1)$ as $\mathfrak{g}$ in Problem 2. There are three different real forms up to isomorphism: (a) $\mathfrak{sl}(2,1;\boldsymbol{R})$, (b) $\mathfrak{su}(2,1;2,1)$, (c) $\mathfrak{su}(2,1;1,1)$, where for $p=1,2$,

$$\mathfrak{su}(2,1;p,1)_s=\{X \in \mathfrak{sl}(2,1)_s : J_{p,1}X+(-1)^s\cdot{}^t\overline{X}J_{p,1}=0\} \qquad (s=0,1),$$

and ${}^tX$ is the transposed of X, and $J_{p,1}=\text{diag}(1,(-1)^p,\sqrt{-1})$, a diagonal matrix.

In cases (b) and (c), because of unitarity of π, ρ must be a highest (or lowest) weight representation. We extend ρ to a complex linear representation of $\mathfrak{g}_{0,c}$, and take $C=\text{diag}(1,1,2)$ and $H=\text{diag}(1,-1,0)$ from $\mathfrak{h}_c$, a Cartan subalgebra of $\mathfrak{g}_{0,c}$.

In case (b), $\mathfrak{g}_0\cong\mathfrak{u}(2)$. We take irreducible unitary representation (ρ,V_0)

with highest weight $\Lambda \in \mathfrak{h}_C^*$. Put $n = \dim V_0 = \Lambda(H) + 1$, and $m = \Lambda(C)$. Then,

Theorem 5. *There exist irreducible unitary extensions* $(=IUEs)$ *of* (ρ, V_0) *if and only if one of the following three conditions holds for* Λ: (i) $n = 1$ *and* $m = -2, 0, 2$; (ii) $n = 2$ *and* $m \in \mathbf{R}, |m| \geq 1$; (iii) $n \geq 3$ *and* $m = \pm(n-1), \pm(n+1)$.

Moreover IUEs are unique up to isomorphism, except the cases $n = 2$ *and* $m = \pm 3$. *In these exceptional cases there exist exactly two IUEs up to isomorphism.*

In case (c), $\mathfrak{g}_0 \cong \mathfrak{u}(1, 1)$. We take a representation ρ in the holomorphic discrete series or its limit, for which the ε should be equal to 1, whereas $\varepsilon = -1$ for the anti-holomorphic case. Let $\Lambda \in \mathfrak{h}_C^*$ be its highest weight. Put $l = -\Lambda(H) \in \mathbf{Z}_{>0}$ and $m = \Lambda(C)$. Then,

Theorem 6. *There exist irreducible unitary extensions of* ρ *if and only if one of the following three conditions holds*: (i) $l = 1$ *and* $m = \pm 1$; (ii) $l = 2$ *and* $m = \pm 2, 0$; (iii) $l \geq 3$ *and* $m = \pm l, \pm(l-2)$.

Moreover IUEs are unique up to isomorphism except the case $l = 2$ *and* $m = 0$. *In this exceptional case there exist exactly two IUEs up to isomorphism.*

By our standard method, we can construct these IUEs explicitely.

Details will appear elsewhere.

References

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