8. Connections for Vector Bundles over Quaternionic Kähler Manifolds

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The purpose of this note is to announce our recent results on quaternionic Kähler manifolds (see Salamon [5] for definition of quaternionic Kähler manifolds). Let M be a $4n$-dimensional connected quaternionic Kähler manifold with the corresponding twistor space $p: Z \rightarrow M$ (cf. [5]). Furthermore, let $\boldsymbol{H}$ be the skew field of quaternions. Then the $Sp(n)\cdot Sp(1)$-module $\wedge^2 \boldsymbol{H}^n$ is a direct sum $N_2' \oplus N_2'' \oplus L_2$ of its irreducible submodules N_2', N_2'', L_2, where N_2' (resp. L_2) is the submodule fixed by $Sp(n)$ (resp. $Sp(1)$) and for $n=1$, we have $N_2''=\{0\}$. Hence, the vector bundle $\wedge^2 T^*M$ is written as a direct sum $A_2' \oplus A_2'' \oplus B_2$ of its holonomy-invariant subbundles in such a way that A_2', A_2'', B_2 correspond to N_2', N_2'', L_2, respectively. Now, let V be a vector bundle over M.

Definition 1. A connection for V is called an *A_2'-connection* (resp. *B_2-connection*) if the corresponding curvature is an End(V)-valued A_2'-form (resp. B_2-form).

First, we have:

Theorem A (cf. [3]). *All A_2'-connections and also all B_2-connections are Yang-Mills connections.*

Let $\rho: Sp(n) \rightarrow GL(2n; \boldsymbol{C})$ be the standard representation of $Sp(n)$. Recall that $Sp(1)=\{h \in \boldsymbol{H} \mid |h|=1\}$. Furthermore, let K' (resp. K'') be the $\boldsymbol{C}$-vector space $\boldsymbol{C}^{2n}$ (resp. $\boldsymbol{C}^2$ $(=\boldsymbol{H})$) endowed with the $Sp(n)$-action (resp. $Sp(1)$-action) defined by

$$Sp(n) \times \boldsymbol{C}^{2n} \ni (g, f) \longrightarrow \rho(g)\cdot f \in \boldsymbol{C}^{2n},$$
$$(\text{resp. } Sp(1) \times \boldsymbol{H} \ni (u, f) \longrightarrow f\cdot u^{-1} \in \boldsymbol{H}).$$

Then the complexification $\boldsymbol{H}^n \otimes_{\boldsymbol{R}} \boldsymbol{C}$ of the $Sp(n)\cdot Sp(1)$-module $\boldsymbol{H}^n$ is naturally identified with $K' \otimes_{\boldsymbol{C}} K''$. Let r be an integer with $r \geqq 2$. Since the submodule $\wedge^r K' \otimes_{\boldsymbol{C}} S^r K''$ of the $Sp(n)\cdot Sp(1)$-module $\wedge^r (K' \otimes_{\boldsymbol{C}} K'')$ $(=\wedge^r (\boldsymbol{H}^n \otimes_{\boldsymbol{R}} \boldsymbol{C}))$ is just $N_r^{\boldsymbol{C}}$ $(=N_r \otimes_{\boldsymbol{R}} \boldsymbol{C})$ for some suitable $Sp(n)\cdot Sp(1)$-module N_r, we have a natural decomposition $\wedge^r \boldsymbol{H}^n = N_r \oplus L_r$ for some complementary $Sp(n)\cdot Sp(1)$-module L_r of N_r in $\wedge^r \boldsymbol{H}^n$ (cf. [3]). Therefore, the vector bundle $\wedge^r T^*M$ is expressed as a direct sum $A_r \oplus B_r$ of subbundles A_r, B_r corresponding to N_r, L_r, respectively. We denote by $\pi^r: \wedge^r T^*M$ $(=A_r \oplus B_r)$ $\rightarrow A_r$ the natural projection to the first factor. Then from a theorem of Salamon [6], one easily obtains the following:

Theorem B (cf. [3]). *Assume that ∇ is a B_2-connection on V. Then the following is an elliptic complex:*

$$0 \longrightarrow \mathcal{E}(V) \xrightarrow{\nabla} \mathcal{E}(V \otimes T^*M) \xrightarrow{d_1} \mathcal{E}(V \otimes A_2) \xrightarrow{d_2} \mathcal{E}(V \otimes A_3) \xrightarrow{d_3} \cdots \xrightarrow{d_{2n-1}} \mathcal{E}(V \otimes A_{2n}) \longrightarrow 0,$$

where $d_i := (\mathrm{id} \otimes \pi^{i+1}) \circ d^{\nabla}$, *and for every vector bundle* W *on* M, *we denote by* $\mathcal{E}(W)$ *the sheaf of germs of* C^∞*-sections of* W.

Now, let ∇ be a B_2-connection on V such that the corresponding holonomy group can be reduced to (a subgroup of) a compact semisimple Lie group G. Then the frame bundle P of V can be regarded as a principal G-bundle. Put $G_P := P \times_\theta G$ and $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$, where $\theta : G \to \mathrm{Aut}(G)$ is the group conjugation and $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is the adjoint representation of G. A global smooth section of G_P is called a *gauge transformation* of P and let $\mathcal{M}$ be the *moduli space* of the B_2-connections on V with holonomy groups in G, where "moduli space" means the space of all such connections modulo gauge transformations of P (see [3] for more details). Then we have the following analogue of a result of Atiyah, Hitchin and Singer [1]:

Theorem C (cf. [3]). *If* ∇ *is an irreducible connection, then the space of infinitesimal deformations of* B_2*-connections at* ∇, *that is, the tangent space of* $\mathcal{M}$ *at* ∇ *is a linear subspace of the first cohomology group of the elliptic complex*:

$$0 \longrightarrow \mathcal{E}(\mathfrak{g}_P) \xrightarrow{\nabla'} \mathcal{E}(\mathfrak{g}_P \otimes T^*M) \xrightarrow{d_1'} \mathcal{E}(\mathfrak{g}_P \otimes A_2) \xrightarrow{d_2'} \mathcal{E}(\mathfrak{g}_P \otimes A_3) \xrightarrow{d_3'} \cdots \xrightarrow{d_{2n-1}'} \mathcal{E}(\mathfrak{g}_P \otimes A_{2n}) \longrightarrow 0,$$

where ∇' *is the connection on* $\mathfrak{g}_P$ *naturally induced by* ∇ *and furthermore, we put* $d_i' := (\mathrm{id} \otimes \pi^{i+1}) \circ d^{\nabla'}$.

For our quaternionic Kähler manifold M, we now define the following:

Definition 2. (i) A pair (E, D_E) of a vector bundle E over M and a B_2-connection D_E on E is called a *Hermitian pair* on M if D_E is a Hermitian connection on E.

(ii) A pair (F, D_F) of a holomorphic vector bundle F over Z and a Hermitian $(1,0)$-connection D_F with Hermitian metric $h(\ ,\ )$ on F is called an *excellent pair* on Z if the following conditions are satisfied:

(a) F is a flat Hermitian vector bundle when restricted to each fibre of $p : Z \to M$. (Hence the real structure $\tau : Z \to Z$ (cf. Nitta and Takeuchi [4] naturally lifts to a bundle automorphism $\tau' : F \to F$.)

(b) Let $\sigma : F \to F^*$ be the bundle map defined fibrewise by

$$F_z \ni f \longmapsto \sigma(f) \in F^*_{\tau(z)} \qquad (z \in Z),$$

where $\sigma(f)(g) := h(f, \tau'(f))$ for each $g \in F_{\tau(z)}$. Then σ is an antiholomorphic bundle automorphism.

We then have the following generalization of a result of Atiyah, Hitchin and Singer [1] (see also Salamon [6], Berard-Bergery and Ochiai [2]):

Theorem D (cf. [3]). *Let* $\mathcal{H}$ (*resp.* $\tilde{\mathcal{H}}$) *be the set of all Hermitian pairs* (*resp. all excellent pairs*) *on* M (*resp.* Z). *Then*

$$\mathcal{H} \ni (E, D_E) \longmapsto (p^*E, p^*D_E) \in \tilde{\mathcal{H}}$$

defines a bijective correspondence: $\mathcal{H} \simeq \tilde{\mathcal{H}}$.

Corollary E (cf. [3]). *Let (F, D_F) be an excellent pair on Z. If M has positive scalar curvature, then F with D_F is a Ricci-flat Einstein Hermitian vector bundle over Z.*

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References

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