41. On the Asymptotic Stability of Solutions of a Second Order Nonlinear Differential Equation

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1. Introduction. In this paper we consider the asymptotic stability of the zero solution of the second order nonlinear differential equation

$$(r(t)x')'+f(t,x,x')x'+p(t)g(x)=0. \tag{1}$$

The stability or attractivity properties of second order nonlinear nonautonomous differential equations are discussed by Burton and Grimmer [2], Baker [1], Hatvani [3], and Yamamoto and Sakata [4], etc. In [3], Hatvani established conditions for the (equi- or uniformly) asymptotic stability of the zero solution of ordinary differential equations and gave an application of his theorem to the equation (1) in the case of $f(t,x,x')\equiv f(t)$.

In the present paper, we investigate the (globally) asymptotic stability, (globally) equi-asymptotic stability, and (globally) uniformly asymptotic stability as well as uniform stability of the zero solution of (1) by applying Hatvani's theorem [3] and one of its extensions [5].

2. Theorems. We consider the equation (1) or the equivalent system

$$x'=y, \qquad y'=-\frac{p(t)}{r(t)}g(x)-\frac{r'(t)+f(t,x,y)}{r(t)}y \tag{2}$$

under the following assumption.

Assumption A. a) p and r are continuously differentiable, positive functions on $\boldsymbol{R}^+=[0,+\infty)$.

(b) $f:\boldsymbol{R}^+\times\boldsymbol{R}^2\to\boldsymbol{R}^+$ is continuous.

(c) $g:\boldsymbol{R}\to\boldsymbol{R}$ is a continuous function such that $xg(x)>0$ $(x\neq 0)$.

We shall use the following notations and definitions.

For $x\in\boldsymbol{R}^n$ and $\varepsilon>0$, let $B_n(x,\varepsilon)=\{y\in\boldsymbol{R}^n:\|y-x\|<\varepsilon\}$. The ε-neighborhood of a set $E\subset\boldsymbol{R}^n$ is the set $B_n(E,\varepsilon)=\{x\in\boldsymbol{R}^n:d(x,E)<\varepsilon\}$, where $d(x,E)=\inf\{\|x-y\|:y\in E\}$ is the distance from $x\in\boldsymbol{R}^n$ to E.

A function a is said to belong to the class $\boldsymbol{K}$ ($a\in\boldsymbol{K}$) if a is a continuous, strictly increasing function on $\boldsymbol{R}^+$ into $\boldsymbol{R}^+$ with $a(0)=0$.

A measurable function $\phi:\boldsymbol{R}^+\to\boldsymbol{R}^+$ is said to be integrally positive (see [3]) if $\int_I\phi(t)dt=+\infty$ on every set $I=\bigcup_{m=1}^{+\infty}[\alpha_m,\beta_m]$ such that $\alpha_m<\beta_m<\alpha_{m+1}$, $\beta_m-\alpha_m\geqq\delta>0$ for $m=1,2,\cdots$. If, in addition, $\alpha_{m+1}-\beta_m\leqq\gamma$ $(m=1,2,\cdots)$ for some constant $\gamma>0$, ϕ is said to be weakly integrally positive (see [3]).

We say that a function $\xi:\boldsymbol{R}^+\to\boldsymbol{R}^+$ belongs to the class $\boldsymbol{F}$ ($\xi\in\boldsymbol{F}$) (see [3]) if there are two measurable functions $\xi_1,\xi_2:\boldsymbol{R}^+\to\boldsymbol{R}^+$ such that ξ_1 is bounded on $\boldsymbol{R}^+$, $\int_0^{+\infty} \xi_2(t)dt < +\infty$ and $\xi(t) = \xi_1(t) + \xi_2(t)$ in $\boldsymbol{R}^+$.

For $h \in \boldsymbol{R}$, let $[h]_+ = \max\{h, 0\}$ and $[h]_- = \max\{-h, 0\}$.

Theorem 1. *Suppose that Assumption* A *and the following conditions hold.*

(i) *There exist positive constants* p_0, p_1, r_0 *and* r_1 *such that* $p_0 \leqq p(t) \leqq p_1$ *and* $r_0 \leqq r(t) \leqq r_1$ *for all* $t \geqq 0$.

(ii) $\int_0^t [r'(s)]_- ds$ *is uniformly continuous in* $\boldsymbol{R}^+$.

(iii) *There exist a constant* $H > 0$ *and continuous functions* $q_1, q_2 : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ *and* $f_1, f_2 : \boldsymbol{R}^2 \to \boldsymbol{R}^+$ *such that*

$$q_1(t) f_1(x, y) \leqq f(t, x, y) \leqq q_2(t) f_2(x, y) \text{ in } \boldsymbol{R}^+ \times \overline{B_2(0, H)}.$$

(iv) $\phi(t) = (p(t)r(t))' + 2mp_0 q_1(t)$ *is integrally positive, where* $m = \min\{f_1(x, y) : (x, y) \in \overline{B_2(0, H)}\}$.

(v) $q_2 \in \boldsymbol{F}$.

Then the zero solution $(x, y) = (0, 0)$ *of* (2) *is uniformly stable and attractive, therefore it is asymptotically stable.*

If, in addition, (iii) *and* (iv) *hold for any constant* $H > 0$ *and*

(vi) $G(x) = \int_0^x g(\lambda) d\lambda \to +\infty$ *as* $|x| \to +\infty$,

then the zero solution of (2) *is globally asymptotically stable.*

Theorem 2. *Suppose that Assumption* A, *the conditions* (i)–(v) *and the following conditions hold.*

(vii) $g(x)$ *satisfies locally a Lipschitz condition.*

(viii) $f(t, x, y)$ *satisfies locally a Lipschitz condition with respect to* (x, y).

Then the zero solution of (2) *is uniformly stable and equi-attractive, therefore it is equi-asymptotically stable.*

If, in addition, (iii) *and* (iv) *hold for any constant* $H > 0$ *and* (vi) *is satisfied, then the zero solution of* (2) *is globally equi-asymptotically stable.*

Corollary 1. *If in* (iv) *the function* ϕ *is weakly integrally positive and* (v) *is replaced by*

(v′) $q_2(t) \leqq c_0$ *in* $\boldsymbol{R}^+$ *for some constant* c_0,

then the statements of Theorems 1 *and* 2 *remain true.*

Theorem 3. *Suppose that Assumption* A, *the conditions* (i), (ii), (iii), (v′) *and the following condition hold.*

(iv′) *There exists a constant* $c_1 > 0$ *such that* $\phi(t) \geqq c_1$ *in* $\boldsymbol{R}_+$.

Then the zero solution of (2) *is uniformly asymptotically stable.*

If, in addition, (iii) *and* (iv′) *hold for any constant* $H > 0$ *and* (vi) *is satisfied, then the zero solution of* (2) *is globally uniformly asymptotically stable.*

We require the following lemmas to prove Theorems 1–3 and Corollary 1. Lemmas 1–4 are the results given by Hatvani [3] and Lemmas 5–7 are obtained in [5] which are extensions of Hatvani's results.

Lemma 1. *Consider the differential equation*

$$x' = X(t, x) \qquad (X(t, 0) = 0), \tag{3}$$

where $X: \boldsymbol{R}^+ \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *is a continuous function. Suppose that there exist continuously differentiable functions* $V, A: \boldsymbol{R}^+ \times \boldsymbol{R}^n \to \boldsymbol{R}$ *and* $W: \boldsymbol{R}^n \to \boldsymbol{R}$ *such that for some constant* $H > 0$, *the following conditions hold.*

(I) *There exist two functions* $a, b \in \boldsymbol{K}$ *such that*

$$a(\|x\|) \leqq V(t, x) \leqq b(\|x\|) \qquad \text{in } \boldsymbol{R}^+ \times B_n(0, H).$$

(II) *There exist a integrally positive function* $\phi: \boldsymbol{R}^+ \to \boldsymbol{R}^+$ *and a continuous function* $U: B_n(0, H) \to \boldsymbol{R}^+$ *such that*

$$V'_{(3)}(t, x) \leqq -\phi(t) U(x) \qquad \text{in } \boldsymbol{R}^+ \times B_n(0, H),$$

where $V'_{(3)}(t, x)$ *implies the derivative of* V *with respect to the equation* (3).

(III) *For every compact set* $M \subset B_n(0, H) \backslash U^{-1}(0)$, *there exists a constant* $\rho = \rho(M) > 0$ *such that* $\overline{B_n^*(M, \rho)} \cap U^{-1}(0) = \varnothing$, *where* $B_n^*(M, \rho) = W^{-1}[B_1(W(M), \rho)] \cap B_n(0, H)$.

(IV) *For every* $t_0 \in \boldsymbol{R}^+$ *and every continuous function* $u: \boldsymbol{R}^+ \to B_n^*(M, \rho)$, $\int_{t_0}^{t} [W'_{(3)}(u(s))]_+ ds$ *is uniformly continuous in* $[t_0, +\infty)$.

(V) *For any real numbers* α_1 *and* α_2 $(0 < \alpha_1 < \alpha_2 < H)$, *there exist positive constants* β, c_2 *and a continuous function* $\psi: \boldsymbol{R}^+ \to \boldsymbol{R}$ *such that* $\int_0^{+\infty} \psi(t) dt = +\infty$, *and for every continuous function* $v: \boldsymbol{R}^+ \to J_n(\alpha_1, \alpha_2)$,

$$A(t, v(t)) \leqq c_2 \quad \text{and} \quad A'_{(3)}(t, v(t)) \geqq \psi(t) \quad \text{in } [t_0, +\infty),$$

where $J_n(\alpha_1, \alpha_2) = \{x \in \overline{B_n(U^{-1}(0), \beta)} : \alpha_1 \leqq \|x\| \leqq \alpha_2\}$.

Then the zero solution of (3) *is uniformly stable and attractive, therefore it is asymptotically stable.*

Lemma 2. *Suppose that the function* ϕ *in* (II) *is only weakly integrally positive and* (V) *is replaced by the following.*

(V′) *For any real numbers* α_1 *and* α_2 $(0 < \alpha_1 < \alpha_2 < H)$, *there exist positive constants* c_3 *and* c_4 *such that*

$$|A(t, x)| \leqq c_3 \quad \text{and} \quad A'_{(3)}(t, x) \geqq c_4 \quad \text{in } \boldsymbol{R}^+ \times J_n(\alpha_1, \alpha_2).$$

Then the statement of Lemma 1 *remains true.*

Lemma 3. *If, in addition to the assumptions in each of Lemmas* 1 *and* 2, X *satisfies a locally Lipschitz condition with respect to* x, *then the zero solution of* (3) *is uniformly stable and equi-attractive, therefore it is equi-asymptotically stable.*

Lemma 4. *Suppose that the function* ϕ *in* (II) *satisfies* $\phi(t) \geqq c_5 > 0$ *in* $\boldsymbol{R}^+$ *for some constant* c_5 *and all the assumptions in Lemma* 1 *except for* (V) *hold. If, in addition,* (V′) *is satisfied, then the zero solution of* (3) *is uniformly asymptotically stable.*

Lemma 5. *Suppose that the following condition is satisfied.*

(I′) *There exist two functions* $a, b \in \boldsymbol{K}$ *such that*

$$a(\|x\|) \leqq V(t, x) \leqq b(\|x\|) \qquad \text{in } \boldsymbol{R}^+ \times \boldsymbol{R}^n$$

and $a(\lambda) \to +\infty$ *as* $\lambda \to +\infty$.

If for any constant $H > 0$, *all the conditions except for* (I) *in each of Lemmas*

1 *and* 2 *hold, then the zero solution of* (3) *is globally asymptotically stable.*

Lemma 6. *If* (I′) *is satisfied and for any constant* $H>0$, *all the conditions except for* (I) *in Lemma* 3 *hold, then the zero solution of* (3) *is globally equi-asymptotically stable.*

Lemma 7. *If* (I′) *is satisfied and for any constant* $H>0$, *all the conditions except for* (I) *in Lemma* 4 *hold, then the zero solution of* (3) *is globally uniformly asymptotically stable.*

To prove Theorems 1–3 and Corollary 1, we use following functions

$$V(t, x, y)=2G(x)+\frac{r(t)}{p(t)}y^2, \tag{4}$$

$$W(x, y)=\frac{1}{2}y^2, \tag{5}$$

$$A(t, x, y)=-r(t)xy, \tag{6}$$

and $U(x, y)=(1/p_1^2)y^2$, as the auxiliary functions in each of the lemmas.

The detailed proof will be published later.

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