39. A Note on a Generalization of a q-series Transformation of Ramanujan

By H. M. SRIVASTAVA*)
Department of Mathematics, University of Victoria,
Victoria, British Columbia V8W 2Y2, Canada

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1987)

It is shown how readily a recent generalization of a q-series transformation of Srinivasa Ramanujan would follow as a limiting case of Heine's transformation for basic hypergeometric series. Several interesting consequences of this general result are also deduced.

For real or complex q, $|q|<1$, let

(1) $$(\lambda;q)_\mu=\prod_{j=0}^{\infty}(1-\lambda q^j)/(1-\lambda q^{\mu+j})$$

for arbitrary λ and μ, so that

(2) $$(\lambda;q)_n=\begin{cases}1, & \text{if } n=0,\\ (1-\lambda)(1-\lambda q)\cdots(1-\lambda q^{n-1}), & \forall n\in\{1,2,3,\cdots\},\end{cases}$$

and

(3) $$(\lambda;q)_\infty=\prod_{j=0}^{\infty}(1-\lambda q^j).$$

The q-series transformation

(4) $$(-bq;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-bq;q)_n}\frac{\lambda^n}{(q;q)_n}=\sum_{n=0}^{\infty}q^{(1/2)n(n+1)}\left(-\frac{\lambda}{b};q\right)_n\frac{b^n}{(q;q)_n}$$

is stated in Chapter 16 of the Second Notebook of Srinivasa Ramanujan [9, Vol. II, p. 194, Entry 9]. A special case of Ramanujan's identity (4) when $b=1$ was posed as an Advanced Problem by Carlitz [5, p. 440, Equation (1)] who, in fact, proved the general case (4) by using Euler's expansion for $(\lambda;q)_n$ as a polynomial in λ (cf. [6, p. 917]). The identity (4) has received considerable attention in several subsequent works (see, for example, [1], [2], and [8]). In particular, in their excellent memoir [1, pp. 9–10] Adiga *et al.* have presented two interesting proofs of (4). It should be remarked in passing that one of their proofs using Heine's transformation [7, p. 306, Equation (79)] iteratively is essentially equivalent to the earlier proof by Andrews [2, p. 105] who deduced (4) as a limiting case of a result attributed to Rogers.

An interesting generalization of Ramanujan's q-series transformation (4) was given recently by Bhargava and Adiga in the form (cf. [4, p. 339, Equation (3)]; see also [3, p. 14, Equation (4*)]):

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*) This research was supported, in part, by the Natural Sciences and Engineering Research Council of Canada under Grant A-7353.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 33A30; Secondary 11B65, 05A30.

$$(5)\qquad (-bq\,;q)_\infty \sum_{n=0}^{\infty} q^{(1/2)n(n+1)} \frac{(-\lambda/a\,;q)_n}{(-bq\,;q)_n} \frac{a^n}{(q\,;q)_n} = (-aq\,;q)_\infty \sum_{n=0}^{\infty} q^{(1/2)n(n+1)} \frac{(-\lambda/b\,;q)_n}{(-aq\,;q)_n} \frac{b^n}{(q\,;q)_n},$$

which would obviously reduce to (4) in the limiting case when $a \to 0$. Replacing λ by λ/q, and setting

$$a = -x/q \quad \text{and} \quad b = -y/q,$$

the identity (5) becomes

$$(6)\qquad \sum_{n=0}^{\infty} q^{(1/2)n(n-1)} \frac{(\lambda/x\,;q)_n}{(y\,;q)_n} \frac{(-x)^n}{(q\,;q)_n} = \frac{(x\,;q)_\infty}{(y\,;q)_\infty} \sum_{n=0}^{\infty} q^{(1/2)n(n-1)} \frac{(\lambda/y\,;q)_n}{(x\,;q)_n} \frac{(-y)^n}{(q\,;q)_n}$$

or equivalently,

$$(7)\qquad {}_1\Phi_1\left[\begin{matrix}\lambda/x\,; \\ y\,;\end{matrix}\; q,\; x\right] = \frac{(x\,;q)_\infty}{(y\,;q)_\infty}\, {}_1\Phi_1\left[\begin{matrix}\lambda/y\,; \\ x\,;\end{matrix}\; q,\; y\right],$$

where ${}_{p+1}\Phi_{p+j}$ $(p, j = 0, 1, 2, \cdots)$ denotes a generalized basic (or q-) hypergeometric series defined by

$$(8)\qquad {}_{p+1}\Phi_{p+j}\left[\begin{matrix}\alpha_1, \cdots, \alpha_{p+1}\,; \\ \beta_1, \cdots, \beta_{p+j}\,;\end{matrix}\; q,\; x\right] = \sum_{n=0}^{\infty} (-1)^{jn} q^{(1/2)jn(n-1)} \frac{(\alpha_1\,;q)_n \cdots (\alpha_{p+1}\,;q)_n}{(\beta_1\,;q)_n \cdots (\beta_{p+j}\,;q)_n} \frac{x^n}{(q\,;q)_n},$$

$$(|x| < \infty \text{ when } j = 1, 2, 3, \cdots, \text{ or } |x| < 1 \text{ when } j = 0).$$

Formula (5) was proven by Bhargava and Adiga [4, pp. 340–341] by making use of Ramanujan's identity (4) and of certain functional relations which they had derived earlier [3, p. 14, Lemma 1] for the left side of (5). With a view to presenting a much shorter and direct proof of the equivalent result (6) or (7), we now recall the aforementioned Heine's transformation [7, p. 306, Equation (79)]

$$(9)\qquad {}_2\Phi_1\left[\begin{matrix}a, b\,; \\ c\,;\end{matrix}\; q, x\right] = \frac{(b\,;q)_\infty (ax\,;q)_\infty}{(c\,;q)_\infty (x\,;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}x, c/b\,; \\ ax\,;\end{matrix}\; q, b\right],$$

which, upon repeated application, yields

$$(10)\qquad {}_2\Phi_1\left[\begin{matrix}a, b\,; \\ c\,;\end{matrix}\; q, x\right] = \frac{(c/b\,;q)_\infty (bx\,;q)_\infty}{(c\,;q)_\infty (x\,;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix}abx/c, b\,; \\ bx\,;\end{matrix}\; q, c/b\right].$$

It is the transformation (10) which was, in fact, employed by Andrews [2, p. 105] as well as Adiga *et al.* [1, pp. 9–11] to deduce Ramanujan's identity (4) as a limiting case. Indeed, as we indicated above, Andrews [2, p. 98, Equation (4.6)] attributed (10) to Rogers, although Heine did give (9) and a relatively more familiar ${}_2\Phi_1$ transformation (cf. [7, p. 325, Theorem XVIII]) which follows readily upon merely iterating Heine's result (9) one more step beyond the transformation (10).

Replacing x by x/b and letting $b \to \infty$ in (10), we have

$$(11)\qquad \sum_{n=0}^{\infty} q^{(1/2)n(n-1)} \frac{(a\,;q)_n}{(c\,;q)_n} \frac{(-x)^n}{(q\,;q)_n} = \frac{(x\,;q)_\infty}{(c\,;q)_\infty} \sum_{n=0}^{\infty} q^{(1/2)n(n-1)} \frac{(ax/c\,;q)_n}{(x\,;q)_n} \frac{(-c)^n}{(q\,;q)_n}.$$

Now set $a = \lambda/x$ and $c = y$ in (11), and we are led immediately to the q-series identity (6) which, in turn, yields the q-hypergeometric form (7) by virtue of the definition (8).

Several consequences of the q-series identity (6) are worthy of note. First of all, if in (6) we set $\lambda=y$, we immediately obtain

$$\sum_{n=0}^{\infty} q^{(1/2)n(n-1)} \frac{(y/x\,;\,q)_n}{(y\,;\,q)_n} \frac{(-x)^n}{(q\,;\,q)_n} = \frac{(x\,;\,q)_\infty}{(y\,;\,q)_\infty} \tag{12}$$

or, equivalently,

$$ {}_1\Phi_1\begin{bmatrix} y/x\,; & q,\, x \\ y\,; & \end{bmatrix} = \frac{(x\,;\,q)_\infty}{(y\,;\,q)_\infty}. \tag{13}$$

Formula (12) reduces, when $x \to 0$ and $y=aq$, to Entry 3 in Chapter 16 of Ramanujan's Second Notebook (cf., e.g., [1, p. 5, Equation (5.1)]).

Next we put $y=q$ and $\lambda=q^2$ in (6), and replace x by qx. We thus find that

$$\sum_{n=0}^{\infty} q^{(1/2)n(n+1)} \frac{(q/x\,;\,q)_n}{\{(q\,;\,q)_n\}^2} (-x)^n = \frac{(qx\,;\,q)_\infty}{(q\,;\,q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{(1/2)n(n+1)}}{(qx\,;\,q)_n}. \tag{14}$$

Finally, we replace q by q^2 in (6), and then set $y=q$ and $\lambda=q^3$. Writing q^2x for x in the resulting identity, we have

$$\sum_{n=0}^{\infty} q^{n(n+1)} \frac{(q/x\,;\,q^2)_n}{(q\,;\,q)_{2n}} (-x)^n = \frac{(q^2x\,;\,q^2)_\infty}{(q\,;\,q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2}}{(q^2x\,;\,q^2)_n}. \tag{15}$$

Each of these last q-series identities (14) and (15) reduces, when $x \to 0$, to a corresponding result given earlier by Adiga *et al.* [1, p. 10, Corollaries (i) and (ii)].

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