33. *Closure-preserving Covers and σ-products*

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Introduction. Recently a number of important results have been obtained concerning properties of spaces which have closure-preserving (c.p.) covers by "nice" (finite, compact, etc.) sets. In particular, if a space X has a c.p. cover by compact sets, then X is metacompact [4, 8]. The reader is referred to [3, 6, 7, 8, 9, 10, 11, 12] for other results.

In 1959, H. H. Corson [2] showed that Σ-products and σ-products played an important role in the study of a number of topological properties and as one result, obtained the following:

Theorem 1. *If X is the σ-product of separable metric spaces X_t, $t \in T$, then X is Lindelöf.*

In this paper we study the σ-product of spaces which have c.p. covers by "nice" sets and obtain results analogous to Theorem 1 above. The reader is referred to [1, 5, 13] for other results of this kind. All spaces will be Hausdorff.

Definition. Let $\{X_t : t \in T\}$ be a family of topological spaces and $p=(p_t) \in \prod X_t$. The *σ-product* of X_t, $t \in T$, having *base point* p, is defined by $X=\{x \in \prod X_t : |\{t \in T : x_t \neq p_t\}| < \aleph_0\}$.

Note that X is a dense subset of $\prod X_t$. In [3] the notion of an ideal of closed subsets was introduced.

Definition. A family $\mathcal{J}$ of closed subsets of a topological spaces X is an *ideal of closed sets* if,

(i) for every finite $\mathcal{J}' \subset \mathcal{J}$, $\bigcup \mathcal{J}' \in \mathcal{J}$ and

(ii) if $J \in \mathcal{J}$ and J' is a closed subset of J, then $J' \in \mathcal{J}$.

Definition. A closure-preserving (c.p.) cover $\mathcal{F} \subset \mathcal{J}$ of a space X is *special* if X has a point finite open cover $\mathcal{U}$ satisfying, for every $F \in \mathcal{F}$ and $x \notin F$, there exists some $U \in \mathcal{U}$ such that $x \in U$ and $U \cap F = \emptyset$.

Properties of spaces having special c.p. $\mathcal{J}$-covers are also studied in [3]. For simplicity we will consider the ideal $\mathcal{C}$ of closed compact sets. The results obtained herein are also true for *any* ideal of closed sets $\mathcal{J}$ which is also finitely productive. The proofs for these ideals follow in an analogous fashion and hence are omitted.

Main Results.

Theorem 2. *Let X be the σ-product of spaces X_t, $t \in T$, where each*

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X_t has a c.p. cover by compact sets. Then X has a c.p. cover by compact sets.

Proof. Let $\mathcal{F}_t$ denote a c.p. cover of X_t by compact sets for each $t \in T$ and let p be a fixed point of X. Define $\mathcal{F}=\{\prod (F_t \cup \{p_t\}) : F_t \in \mathcal{F}_t \cup \{\phi\}, |\{t \in T : F_t \neq \phi\}| < \aleph_0, t \in T\}$. We assert that $\mathcal{F}$ is a c.p. cover of X by compact sets. It is easy to see that $\mathcal{F}$ covers X and that each member of $\mathcal{F}$ is compact.

Now let $\phi \neq \mathcal{F}' \subseteq \mathcal{F}$ and $x \in X - \cup \mathcal{F}'$. Define $T(x)=\{t \in T : x_t \neq p_t\}$ so that $T(x)$ is finite. Since $p \in \bigcup \mathcal{F}'$, $T(x) \neq \phi$. For each $F \in \mathcal{F}'$, $x \notin F$ so that there exists some $t(F) \in T(x)$ such that $x_{t(F)} \notin F_{t(F)}$. Now for each $s \in T(x)$, define $U_s = X_s - \bigcup \{F_s : s = t(F) \text{ for some } F \in \mathcal{F}'\}$, so that U_s is open in X_s and contains x_s. Define $U = \prod U_t$ where $U_t = X_t$ for $t \notin T(x)$. It is easy to see that $x \in U$ and $U \cap (\bigcup \mathcal{F}') = \phi$. Therefore $\mathcal{F}$ is the desired c.p. cover of X.

Remark. Yakovlev [13] has shown that the σ-product of compact metric spaces has a c.p. cover by compact metric spaces.

Theorem 3. *Let X be the σ-product of spaces X_t, $t \in T$, such that each X_t has a special c.p. cover by compact sets. Then X has a special c.p. cover by compact sets.*

Proof. The proof follows in the same fashion as that for Theorem 2 above with the following modification. For each $t \in T$, let $\mathcal{U}_t=\{U(t, s) : s \in S_t\}$ be a point finite open cover of X_t which is associated with $\mathcal{F}_t$ above. Define

$$V(t, s)=\{x \in X : x_t \in U(t, s)-\{p_t\}\} \quad \text{and}$$
$$\mathcal{V}=\{X\} \cup \{V(t, s) : s \in S_t, t \in T\}.$$

It is easy to verify that $\mathcal{V}$ is a point finite open cover of X. Define $\mathcal{F}$ as in Theorem 2 above. For $\phi \neq \mathcal{F}' \subseteq \mathcal{F}$ and $x \notin \bigcup \mathcal{F}'$, we can construct $V \in \mathcal{V}$ (as in Theorem 2) so that $x \in V$ and $V \cap (\bigcup \mathcal{F}')=\phi$. Therefore X has a special c.p. cover by compact sets.

The next result is analogous to Theorem 1 above.

Theorem 4. *Let X be the σ-product of spaces X_t, $t \in T$, where each X_t is Lindelöf and has a c.p. cover by compact sets. Then X is Lindelöf.*

Proof. Let $T(x)=\{t \in T : x_t \neq p_t\}$ as before. Note that $X=\bigcup_n Y_n$, where $Y_n=\{x \in X : |T(x)| \leqslant n\}$. Moreover, each Y_n is the continuous image of a closed subset E_n of $Z_n = Y_1 \times \cdots \times Y_1$ (n times). Indeed, put

$$E_n = Z_n - \bigcup_t \bigcup_i \bigcup_{j \neq i} \{z \in Z_n : z_{i,t} \neq z_{j,t}\},$$

and define $f_n : E_n \to Y_n$ by $f_n(z)=y$, where $y_t = z_{i,t}$ if $t \in T(z_i)$ for some $i \leqslant n$ and $y_t = p_t$ otherwise. It remains to show that Z_n is Lindelöf for any n. First, we shall show that $Z_1 = Y_1$ is a Lindelöf space. Let U be an open set in Y_1 containing p. Then there are open sets $U_{t_1}, \cdots, U_{t_k}$ in $X_{t_1}, \cdots, X_{t_k}$ respectively so that

$$p \in \{y \in Y_1 : y_{t_1} \in U_{t_1}, \cdots, y_{t_k} \in U_{t_k}\} \subset U.$$

Hence $Y_1 - U \subset \bigcup_{i=1}^k \{y \in Y_1 : y_{t_i} \in X_{t_i} - U_{t_i}\}$.

Therefore, $Y_1 - U$ is a closed subset of the finite union of Lindelöf sets, and hence is Lindelöf. By a similar argument, the Lindelöf property of

Z_n follows by induction.

Remark. Arhangel'skiĭ and Rančin [1] have shown that the σ-product of σ-compact spaces is σ-compact and hence Lindelöf.

References

[1] A. V. Arhangel'skiĭ and D. V. Rančin: Everywhere dense subspaces of topological products and properties associated with final compactness. Vestnik Moskov. Univ., Ser. I, Mat. Meh., no. 6, 21–28 (1982) (in Russian); Moskov. Univ. Math. Bull., **37**, no. 6, 21–29 (1982) (English translation).

[2] H. H. Corson: Normality in subsets of product spaces. Amer. J. Math., **81**, 785–796 (1959).

[3] H. Junnila, J. C. Smith, and R. Telgársky: Closure-preserving covers by small sets. Top. Appl., **23**, 237–262 (1986).

[4] Y. Katuta: On spaces which admit closure-preserving covers by compact sets. Proc. Japan Acad., **50**, 826–828 (1974).

[5] L. V. Nahmanson and N. N. Yakovlev: On bicompacta lying in σ-products. Comment. Math. Univ. Carolin., **22**, 705–719 (1981) (in Russian).

[6] H. B. Potoczny: A nonparacompact space which admits a closure-preserving cover of compact sets. Proc. Amer. Math. Soc., **32**, 309–311 (1972).

[7] ——: Closure-preserving families of compact sets. Gen. Topology Appl., **3**, 243–248 (1973).

[8] H. Potoczny and H. Junnila: Closure-preserving families and meta-compactness. Proc. Amer. Math. Soc., **53**, 523–529 (1975).

[9] R. Telgársky: Closure-preserving covers. Fund. Math., **85**, 165–175 (1974).

[10] ——: Spaces defined by topological games. ibid., **88**, 193–223 (1975).

[11] R. Telgársky and Y. Yajima: On order locally finite and closure-preserving covers. ibid., **109**, 211–216 (1980).

[12] Y. Yajima: On spaces which have a closure-preserving cover by finite sets. Pacific J. Math., **69**, 571–578 (1977).

[13] N. N. Yakovlev: On bicompacta in Σ-products and related spaces. Comment. Math. Univ. Carolin., **21**, 263–283 (1980).