3. Propagation of Chaos for the Two Dimensional Navier-Stokes Equation

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In this paper we establish a rigorous derivation of the two dimensional vorticity equation associated with the Navier-Stokes equation from a many particle system as a propagation of chaos.

It is well known that an incompressible and viscous two dimensional fluid, under the action of an external conservative field is described by the following evolution equations

$$(1)\qquad \nabla_t v(t,z)+(u\cdot\nabla)v(t,z)-\nu\Delta v(t,z)=0,$$

$$(2)\qquad \begin{cases} v(t,z)=\operatorname{curl} u(t,z)=\nabla_x u_2-\nabla_y u_1, \\ \nabla\cdot u=0, \qquad z=(x,y)\in R^n \end{cases}$$

where $u=(u_1,u_2)\in R^2$ is the velocity field and $\nabla_x=\partial/\partial x$, $\nabla_y=\partial/\partial y$, $\nabla=(\nabla_x,\nabla_y)$. $\nu>0$ denotes the viscosity constant. Introducing the operator $\nabla^\perp=(\nabla_y,-\nabla_x)$, by virtue of $\nabla\cdot u=0$, one obtains

$$(3)\qquad u(t,z)=\int_{R^2}(\nabla^\perp G)(z-z')v(t,z')dz',$$

where $G(z)=-(2\pi)^{-1}\log|z|$ is the fundamental solution of the Poisson equation. By means of (3), (1) turns to be a closed equation and is nothing but a McKean's type non-linear equation (see H. P. McKean [1]). Hence a probabilistic treatment for the equation (1) is possible. Such an observation for the two dimensional Navier-Stokes equation was made by Marchioro-Pulvirenti in [2]. We shall discuss "a propagation of chaos for the equation (1)".

Let $\{Z_t\}$ denote the McKean process associated with (1);

$$(4)\qquad dZ_t=\sigma dB_t+E[(\nabla^\perp G)(Z_t-Z'_t)\,|\,Z_t], \qquad \sigma=\sqrt{2\pi}$$

where $B_\cdot$ is a 2-dimensional Brownian motion and $Z'_\cdot$ is an independent copy of $Z_\cdot$.

The n particle system associated with (1) are described by the following S.D.E.s,

$$(5)\qquad dZ_t^i=\sigma dB_t^i+(n-1)^{-1}\sum_{\substack{j\neq i\\ j=1}}^{n}(\nabla^\perp G)(Z_t^i-Z_t^j)dt, \qquad 1\leqq i\leqq n,$$

where $(B_\cdot^1,\cdots,B_\cdot^n)$ is a $2n$-dimensional Brownian motion. Since the coefficients of (4) have singularities at $\mathcal{N}=\bigcup_{i\neq j}\{z=(z_1,\cdots,z_n)\in R^n,\ z_i\neq z_j\}$, it is not trivial to see that the solution of (4) defines a conservative diffusion process on R^{2n}. However, if it starts out side of $\mathcal{N}$, it can be shown that this diffusion process does not hit $\mathcal{N}$ (see Osada [4]).

Let us introduce a

Definition. If E is a separable metric space, a sequence of symmetric probabilities m_n on E^n is said to be m-chaotic for a probability m on E, if for $f_1, \cdots, f_k$, continuous bounded functions on E,

$$\lim_{n\to\infty} \langle m_n, f_1\otimes\cdots\otimes f_k\otimes 1\otimes\cdots\otimes 1\rangle = \prod_{i=1}^{k} \langle m, f_i\rangle,$$

holds. In the following $M(E)$ will denote the set of probabilities on E. One can show (see Tanaka [6], Sznitman [5]) that being m-chaotic is equivalent to the convergence in law of $X_n = n^{-1}\sum_{i=1}^{n}\delta_{X_i}$, (which is an $M(E)$-valued random variable defined on (E^n, m_n), X_i are the canonical coordinates on E^n), towards the non-random m.

In the following, $\mathcal{C}$ will denote $C([0,\infty)\to R^2)$. Let $\{Z_\cdot^n = (Z_\cdot^1, \cdots, Z_\cdot^n)\}$ (resp. $\{Z_\cdot\}$) be the solution of (5) ((4)) with initial distribution $\psi_n(z_1, \cdots, z_n)$ $dz_1\cdots dz_n$ ($\psi(z)dz$) and $P_n(P)$ be the probability measure on $\mathcal{C}^n(\mathcal{C})$ induced by $\{Z_\cdot^n\}$ ($\{Z_\cdot\}$). Now we state our main result:

Theorem. *Assume $\psi_n dz_1\cdots dz_n$ is ψdz-chaotic and*

$$(6)\qquad \lim_{n\to\infty}\sup_{k\geqq n}\left\|\int_{R^{2k-2i}} \psi_n dz_{i+1}\cdots dz_k\right\|_{L^\infty(R^{2i})} < \infty \qquad (i=1,2,4).$$

Then there exists a positive constant ν_0 such that, if $\nu > \nu_0$, then P_n is P-chaotic.

It is convenient to state the theorem in another way. Let $\bar{Z}_n = \mathcal{C}_n^{-1}\sum_{I_n}\delta(Z_\cdot^{i_1}, \cdots, Z_\cdot^{i_6})$ ($I_n = \{(i_1, \cdots, i_6);\ 1\leqq i_k\leqq n,\ i_k\neq i_j \text{ if } k\neq j\}$) and $\bar{P}_n = \bar{Z}_n\circ P_n \in M(M(\mathcal{C}^6))$. $\mathcal{C}_n$ denotes the normalized constant. Put $\bar{P} = \delta_{P^6}\in M(M(\mathcal{C}^6))$. Then, as we explained above, Theorem is equivalent to

Theorem′. *Assume $\{\psi_n dz_1\cdots dz_n\}$ and ψ satisfy the same conditions of Theorem. Then $\lim_{n\to\infty}\bar{P}_n = \bar{P}$ in $M(M(\mathcal{C}^6))$.*

Now we proceed to a sketch of the proof.

1. Let us first show the tightness of $\{P_n\}$. Let $c_{ij}(s,x)$ $(i,j=1,\cdots,n)$ be bounded measurable functions. A differential operator

$$A = \alpha\Delta + \sum_{i,j=1}^{n} (\nabla_i c_{ij})\nabla_j$$

on R^n (α is a constant, $\nabla_i = \partial/\partial x_i$) is said to be of class $\mathcal{G}(n,\alpha,\beta)$ if

$$(7)\qquad \int_{R^n}\sum_{i,j=1}^{n} c_{ij}\nabla_i\nabla_j f dx = 0, \qquad \text{for any } f(x)\in C_0^2(R^n),$$

$$(8)\qquad |c_{ij}|\leqq \beta/n.$$

We call A is of class $\mathcal{G}_0(n,\alpha,\beta)$ if $A\in\mathcal{G}(n,\alpha,\beta)$ and the coefficients are smooth.

Lemma 1. *Let $A\in\mathcal{G}_0(n,\alpha,\beta)$. Then the fundamental solution $p = p(s,x,t,y)$ of $\nabla_s + A$ satisfies*

$$(9)\qquad \sum_{i=1}^{n}\int_{R^n} |x_i - y_i|^q p(s,x,t,y)dy \leqq C_1 n|t-s|^{q/2}$$

for $0<s<t<\infty$, any $x\in R^n$ with a positive constant C_1 depending only on α, β and q.

(See Osada [3] for the proof.) Let L_n be the generator of (5). Then

$$(10)\qquad L_n \in \mathcal{G}(2n, \nu, 2).$$

(See Osada [4] for the proof.) By (9) and (10), we have

(11) $$\sum_{i=1}^{n} E_{P_n}(|Z_t^i - Z_s^i|^4) \leqq C_2 n |t-s|^2$$

where C_2 is independent of the dimension n. Taking into account of symmetry of $(Z_\cdot^1, \cdots, Z_\cdot^n)$, we can conclude from (11) that $\{P_n\}$ is tight.

2. Next we state the uniqueness result for weak solutions of (1). A family of probability measures $\{v_t(dz)\}$ $(0 \leqq t < \infty)$ on R^2 is called a weak solution of (1) with initial condition v_0 if

(12) $$\int_0^t \int_{R^4} |z_1 - z_2|^{-1} v_s(dz_1) v_s(dz_2) ds < \infty,$$

(13) $$\langle v_s, f(s,\cdot)\rangle\Big|_{s=0}^{s=t} - \int_0^t \langle v_s, (\nabla_s + \nu\Delta) f(s,\cdot)\rangle ds - \int_0^t \int_{R^4} (\nabla^\perp G)(z_1 - z_2)\cdot(\nabla f)(s, z_1) v_s(dz_1) v_s(dz_2) ds = 0$$

for all $f(t,z) \in C_0^2([0,\infty)\times R^2)$.

Proposition 1. *Suppose* $\{v_t(dz)\}$ *is a weak solution of* (1) *with initial condition* $v_0(dz) = v(z)dz$ *and that* $v(z) \in L^\infty(R^n)$ *and that* $v_t(dz)$ *has a density* $v(t,z)$ *for* a.e. t *such that*

(14) $$\int_0^t \left(\int_{R^2} v(s,z)^2 dz\right)\left(\int_{R^2} |v(s,z)|^3 dz\right) ds < \infty.$$

Then $\{v_t(dz)\}$ *is unique.*

3. Let $\bar{P}$ be an arbitrary limit point of $\{\bar{P}_n\}$. It can be easily seen that $\bar{P}(\{m \in M(C^6);\ {}^\exists\tilde{m} \in M(C),\ m = \tilde{m}\otimes\cdots\otimes\tilde{m}\}) = 1$.

Proposition 2. *For* $\bar{P}$ a.e. $m \in M(C^6)$, $\tilde{m} \in M(C)$ *is a weak solution of* (1).

To show Proposition 2, we consider a function $H^{+(-)}$ on $M(C^6)$,

$$H^{+(-)}(m) = \left\langle m, \left[\sum_{i=1}^{2}\left\{f(t, Z_t^i) - f(s, Z_s^i) - \int_s^t (\nabla_u + \nu\Delta) f(u, Z_u^i) du\right\} - \int_s^t h^{+(-)}(u, Z_u^1, Z_u^2) du\right]\right\rangle$$

where for $f \in C_0^2([0,\infty)\times R^2)$, h^+ (resp. h^-) is a upper (lower) semicontinuous version of

$$(\nabla^\perp G)(z_1 - z_2)\cdot\{(\nabla f)(t, z_1) - (\nabla f)(t, z_2)\}.$$

It should be noted that H^+ (resp. H^-) is a bounded upper (lower) semicontinuous function on $M(C^6)$. Hence we have

Lemma 2. *For* $\bar{P}$ a.e. $m \in M(C^6)$,

(15) $$H^+(m) \geqq 0 \quad \textit{and} \quad H^-(m) \leqq 0.$$

By using Ito's formula for $r(z) = |z|$, we have

Lemma 3. *There exists a positive constant* ν_0 *such that, if* $\nu \geqq \nu_0$, *then*

(16) $$\sup_n E_{P_n}\left(\int_0^t |Z_s^1 - Z_s^2|^{-1} ds\right) < \infty.$$

By (16) we have, for $\bar{P}$ a.e. m,

(17) $$\left\langle m, \int_0^t |Z_s^1 - Z_s^2|^{-1} ds\right\rangle < \infty$$

and

(18) $$H^+(m) = H^-(m) = 0.$$

On account of the symmetry of $Z^1_\cdot$ and $Z^2_\cdot$, (13) follows from (17) and (18), which completes the proof of Proposition 2.

4. The final step is

Proposition 3. *There exists a positive constant* ν_0 *such that if* $\nu \geqq \nu_0$, *then, for* P a.e. $m \in M(C^6)$, $\tilde{m}$ *has a density* $m_t(z)dz$ *for* a.e. $t>0$ *satisfying* (14).

Let $g_h(z)=(2\pi h)^{-3}\exp(-|z|^2/h)$, $z=(z_1, z_2, z_3)\in R^6$. It is not difficult to see that Proposition 3 follows from

Lemma 4.

$$\sup_{h>0} E_P\left(\left\langle m, \int_0^t g_h(Z^1_s-Z^2_s, Z^3_s-Z^4_s, Z^3_s-Z^5_s)ds\right\rangle\right)<\infty. \tag{19}$$

We can reduce (19) to

$$\varlimsup_{n}\sup_{h>0} E_{P_n}\left(\left\langle\int_0^t g_h(Z^1_s-Z^2_s, Z^3_s-Z^4_s, Z^3_s-Z^5_s)ds\right)<\infty. \tag{20}$$

The key point of the proof of (20) is to show

Lemma 5.

$$\varlimsup_{n} E_{P_n}\left(\int_0^t (|Z^1_s-Z^2_s|^2+|Z^1_s-Z^3_s|^2+|Z^4_s-Z^5_s|^2)^{-5/2}|Z^i_s-Z^6_s|^{-1}\,ds\right)<\infty \quad (i=1, 2, 4). \tag{21}$$

The details of the proof will be given elsewhere.

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