21. On Polarized Manifolds of Sectional Genus Two

By Takao FUJITA
Department of Mathematics, College of Arts and Sciences,
University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 12, 1986)

Let L be an ample line bundle on a compact complex manifold M of dimension n. Then the *sectional genus* of the polarized manifold (M, L) is given by the formula

$$2g(M, L)-2=(K+(n-1)L)L^{n-1},$$

where K is the canonical bundle of M. We have a satisfactory classification theory of polarized manifolds with $g(M, L)\leqq 1$ (see [1]). In this note we study the case $g(M, L)=2$. Details and proofs will be published elsewhere.

Definition. Let (M, L) be a polarized manifold and let p be a point on M. Let $\pi: M'\to M$ be the blowing-up at p and set $L'=\pi^*L-E$, where E is the exceptional divisor. If L' is ample, the polarized manifold (M', L') is called the *simple blowing-up* of (M, L) at p. Note that $g(M', L')=g(M, L)$ and $(L')^n=L^n-1$ in this case.

Theorem A. *Let (M, L) be a polarized manifold with $g(M, L)=2$, $n\geqq 3$ and $d=L^n>0$. Then one of the following conditions is satisfied:*

1) $K=(3-n)L$ *in* Pic(M) *and* $d=1$.

2) *M is a double covering of $\boldsymbol{P}^n$ with branch locus being a smooth hypersurface of degree* 6, *and L is the pull-back of* $\mathcal{O}(1)$. $d=2$.

2′) *(M, L) is a simple blowing-up of another polarized manifold (M_0, L_0) of the above type* 2). $d=1$ *and* $n=3$.

3) *There is a vector bundle $\mathcal{E}$ on a smooth surface S such that $M\simeq \boldsymbol{P}_S(\mathcal{E})$ and L is the tautological line bundle* $\mathcal{O}(1)$.

4) *There is a vector bundle $\mathcal{E}$ on a smooth curve C of genus two such that $M\simeq \boldsymbol{P}_C(\mathcal{E})$ and $L=\mathcal{O}(1)$.*

5) *There is a surjective morphism $f: M\to C$ onto a smooth curve C such that any fiber F of f is a hyperquadric in $\boldsymbol{P}^n$ and $L_F=\mathcal{O}_F(1)$.*

For a proof, we use the polarized version of Mori-type theory in [1]. The above conditions 2), 2′) and 4) are descriptive enough, so we will study the case 1), 3) and 5) in the sequel.

Theorem B. *Let (M, L) be a polarized manifold as in Theorem* A, 5). *Then there is a vector bundle $\mathcal{E}$ on C such that M is embedded in $P=\boldsymbol{P}_C(\mathcal{E})$ as a divisor, L is the restriction of the tautological line bundle H on P and $M\in|2H+\pi^*B|$ for some $B\in$ Pic(C), where π is the projection $P\to C$. Moreover $h^1(C, \mathcal{O}_C)=0$ or 1. Set $b=\deg(B)$. Then:*

b0) *If $C\simeq \boldsymbol{P}^1$, then one of the following conditions is valid.*

b0-1) $d=1$, $b=5$ *and* $\mathcal{E}\simeq\mathcal{O}_C(-1,-1,0,0)$. *This means that* $\mathcal{E}$ *is the direct sum of* $\mathcal{O}_C(-1)$, $\mathcal{O}_C(-1)$, $\mathcal{O}_C$ *and* $\mathcal{O}_C$.
b0-2) $d=2$, $b=4$ *and* $\mathcal{E}\simeq\mathcal{O}_C(-1,0,0,0)$.
b0-3) $d=3$, $b=3$ *and* $\mathcal{E}\simeq\mathcal{O}_C(0,0,0,0)$. $\mathrm{Bs}|L|=\phi$ *and* $|L|$ *makes* M *a triple covering of* $\boldsymbol{P}^n$.
b0-3*) $d=3$, $b=3$ *and* $\mathcal{E}\simeq\mathcal{O}_C(-1,0,0,1)$. $\mathrm{Bs}|L|$ *is a point.*
b0-4) $d=4$, $b=2$ *and* $\mathcal{E}\simeq\mathcal{O}_C(0,0,0,1)$. *M is the normalization of a hypersurface of degree four in* $\boldsymbol{P}^4$ *which has double points along a line.*
b0-5) $d=5$, $b=1$ *and* $\mathcal{E}\simeq\mathcal{O}_C(0,0,1,1)$.
b0-6) $d=6$, $b=0$ *and* $\mathcal{E}\simeq\mathcal{O}_C(0,1,1,1)$. *M is a double covering of* $\boldsymbol{P}^1\times\boldsymbol{P}^2$ *with branch locus being a smooth divisor of bidegree* (2, 2).
b0-7) $d=7$, $b=-1$ *and* $\mathcal{E}\simeq\mathcal{O}_C(1,1,1,1)$. *M is the blowing-up of* $\boldsymbol{P}^3$ *with center being a smooth complete intersection of two hyperquadrics.*
b0-8) $d=8$, $b=-2$ *and* $\mathcal{E}\simeq\mathcal{O}_C(1,1,1,2)$. *M is the blowing-up of a smooth hyperquadric in* $\boldsymbol{P}^4$ *along a smooth conic curve.*
b0-8$^\#$) $d=8$, $b=-2$ *and* $\mathcal{E}\simeq\mathcal{O}_C(1,1,1,1,1)$. *M is the product* $\boldsymbol{P}^1\times Q$, Q *being a hyperquadric in* $\boldsymbol{P}^4$.
b0-9) $d=9$, $b=-3$ *and* $\mathcal{E}\simeq\mathcal{O}_C(1,1,2,2)$. *M is the product* $\boldsymbol{P}^1\times\Sigma_1$, Σ_1 *being the blowing-up of* $\boldsymbol{P}^2$ *at a point.*

b1) *If C is an elliptic curve, then one of the following conditions is valid.*
b1-1) $d=1$, $b=1$ *and* $\deg(\det(\mathcal{E}))=0$. *Moreover* $\deg(Q)\geqq 0$ *for any quotient bundle* Q *of rank one of* $\mathcal{E}$.
b1-2) $d=2$, $b=0$ *and* $\deg(\det(\mathcal{E}))=1$. *Moreover H is nef.*
b1-3) $d=3$, $b=-1$ *and* $\deg(\det(\mathcal{E}))=2$. *Moreover H is ample and* $n=3$.

Remark 1. In the above case L is very ample if and only if $d\geqq 5$.

Remark 2. Polarized manifolds of the above type b1-1) and b1-2) do really exist in arbitrary dimension.

Proposition C. *Let* (M, L) *be a polarized manifold as in Theorem* A, 1). *Then* $H^i(M, tL)=0$ *for any* $t\in\boldsymbol{Z}$, $0<i<n$. *Moreover* $h^0(M, L)\leqq n$.
c1) $h^0(M, L)=n$ *if and only if* (M, L) *is a hypersurface of weighted degree* 10 *in the weighted projective space* $\boldsymbol{P}(5, 2, 1, \cdots, 1)$.
c2) $h^0(M, L)=n-1$ *if and only if* (M, L) *is a weighted complete intersection of type* (6, 6) *in the weighted projective space* $\boldsymbol{P}(3, 3, 2, 2, 1, \cdots, 1)$.
c3) $h^0(M, L)\geqq 1$ *and* $\tau=\sharp\,\{$*torsion part of* $\mathrm{Pic}(M)\}\leqq 5$ *if* $n=3$. *Moreover*
$\tau=5$ *if and only if* $\pi_1(M)\simeq\boldsymbol{Z}/5\boldsymbol{Z}$ *and the universal covering* $\tilde{M}$ *of M is a hypersurface of degree five in* $\boldsymbol{P}^4$ *and*
$\tau=4$ *if and only if* $\pi_1(M)\simeq\boldsymbol{Z}/4\boldsymbol{Z}$ *and* $\tilde{M}$ *is a weighted complete intersection of type* (4, 4) *in the weighted projective space* $\boldsymbol{P}(2, 2, 1, 1, 1, 1)$.

Remark. At present, we have no example with $n\geqq 4$ and $h^0(M, L)<n-1$, nor with $n=3$, $h^0(M, L)=1$ and $\tau\leqq 3$.

Now we consider the case Theorem A, 3). $A=\det(\mathcal{E})$ turns out to be an ample line bundle with $g(S, A)=2$. So we first establish the following

Theorem D. *Let* (S, A) *be a polarized surface with* $g(S, A)=2$. *Then* *one of the following conditions is satisfied.*

d0) *(S, A) is a simple blowing-up of another polarized surface.*

d1) *The canonical bundle K of S is numerically equivalent to A. $d=1$ in this case.*

d2) *K is numerically trivial and $d=2$.*

d3) *S is a P^1-bundle over an elliptic curve C and $AF=3$ for any fiber F of $S \to C$. $d=3$.*

d4) *S is a P^1-bundle over an elliptic curve C and $AF=2$ for any fiber F of $S \to C$. $d=4$.*

d5) *S is the blowing-up at a point on a P^1-bundle over an elliptic curve C. $AF=5$ for any general fiber F of $S \to C$ and $AE=2$ for the exceptional curve E. $d=1$.*

$d6_0$) *$S \simeq P^1 \times P^1$ and $A=\mathcal{O}(2, 3)$. $d=12$.*

$d6_1$) *$S \simeq \Sigma_1$, the blowing-up of P^2 at a point. $A=4H-2E$, where H is the pull-back of $\mathcal{O}_{P^2}(1)$ and E is the exceptional curve. $d=12$ in this case.*

$d6_2$) *$S \simeq \Sigma_2 = P_{P^1}(\mathcal{O}(0, 2))$ and $A=2H_\sigma + H_\beta$, where H_β is the pull-back of $\mathcal{O}_{P^1}(1)$ and H_σ is the tautological line bundle. $d=12$ in this case.*

d7) *$-K$ is ample, $K^2=1$ and $A=-2K$. In this case S is the blowing-up of P^2 at eight points and $d=4$.*

d8) *There are two points P_1, P_2 on a polarized surface (S_0, L_0) of the above type* d7) *such that S is the blowing-up of S_0 and $L=-3K+E_1+E_2$, where E_i is the exceptional curve over p_i. $d=1$ in this case.*

d9) *S is a P^1-bundle over a smooth curve C of genus two and $AF=1$ for any fiber F of $S \to C$.*

Theorem E. *Let (M, L), S, $\mathcal{E}$ and $A=\det(\mathcal{E})$ be as in Theorem* A, 3). *Then one of the following conditions is satisfied.*

e1) *There is a smooth curve C of genus two and a point p on C such that M is isomorphic to the symmetric product $C \times C \times \cdots \times C/\mathrm{S}_n$ and L is numerically equivalent to the divisor $(D_1 + \cdots + D_n)/\mathrm{S}_n$, where $D_j = \pi_j^*[p]$ with π_j being the j-th projection $C \times \cdots \times C \to C$. In this case S is the Jacobian variety of C and $d=1$. (S, A) is of the type* d2).

e2) *There is an indecomposable vector bundle $\mathcal{F}$ on an elliptic-curve C such that $S \simeq P_C(\mathcal{F})$ and $H^2 = c_1(\mathcal{F}) = 1$ for the tautological line bundle H on S. There is an exact sequence $0 \to \mathcal{O}_S[2H+B_1] \to \mathcal{E} \to \mathcal{O}_S[H+B_2] \to 0$ for some line bundles B_1, B_2 coming from* $\mathrm{Pic}(C)$. *Moreover,* $(d, \deg(B_1), \deg(B_2)) = (1, -2, 1)$ *or* $(2, -1, 0)$. *(S, A) is of the type* d3).

$e2^\#$) *There are $\mathcal{F}$ and C as in* e2) *such that $S \simeq P_C(\mathcal{F})$. Moreover $\mathcal{E} \simeq \pi^* \mathcal{G} \otimes H$ for some vector bundle $\mathcal{G}$ on C with* $\mathrm{rank}(\mathcal{G})=3$, $c_1(\mathcal{G})=-1$. *In this case $n=4$, $d=2$ and (S, A) is of the type* d3).

e3) *There are vector bundles $\mathcal{F}$, $\mathcal{G}$ of rank two on an elliptic curve C such that $S \simeq P_C(\mathcal{F})$ and $\mathcal{E} \simeq \pi^* \mathcal{G} \otimes H$, where H is the tautological line bundle on S. Moreover $(c_1(\mathcal{F}), c_1(\mathcal{G})) = (0, 1)$ or $(1, 0)$. $n=3$, $d=3$ and (S, A) is of the type* d4) *in this case.*

e4) *(S, A) is of the type* d7) *and $\mathcal{E} = [-K_S] \oplus [-K_S]$. So $M \simeq S \times P^1$ and $d=3$.*

e5) *$S\simeq \boldsymbol{P}^1\times \boldsymbol{P}^1$ and $\mathcal{E}=\mathcal{O}(1,1)\oplus\mathcal{O}(1,2)$. $d=9$ and $n=3$.*
e6) *(S, A) is of the type* $\mathrm{d}6_1$) *and $\mathcal{E}=[2H-E]\oplus[2H-E]$. In this case $M \simeq \Sigma_1\times \boldsymbol{P}^1$ and $d=9$.*

Remark. The (M, L) in e5) and e6) are the same as that in b0-9). There are three different ways to describe (M, L) because Pic (M) is of rank three. On the other hand, the (M, L) in e4) is different from those in b0-3) and b0-3*) because $\mathrm{Bs}|L|$ is a curve in case e4). The (M, L) in e3) satisfies also the condition b1-3). However, there are examples which satisfy b1-3), but not e3).

Reference

[1] T. Fujita: On polarized manifolds whose adjoint bundles are not nef (preprint).