117. A Note on the Approximate Functional Equation for $\zeta^2(s)$. III

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1. Let $\mathcal{E}_2(s, \alpha t/2\pi)$ be the error-term in the approximate functional equation for $\zeta^2(s)$, i.e.

$$\mathcal{E}_2(s, \alpha t/2\pi)=\zeta^2(s)-\sum_{n\leqq \alpha t/2\pi}{}' d(n)n^{-s}-\chi^2(s)\sum_{n\leqq t/2\pi\alpha}{}' d(n)n^{s-1},$$

where $\chi(s)$ is the Γ-factor in the functional equation for $\zeta(s)$, and the prime indicates that $d(\alpha t/2\pi)$ and $d(t/2\pi\alpha)$ are halved; naturally we use the convention that $d(x)=0$ if x is not an integer.

The problem of finding an asymptotic expansion for $\mathcal{E}_2(s, \alpha t/2\pi)$ has been solved in our former note [2] when $\alpha=1$ the symmetric case. Here we shall show a solution for the non-symmetric case where α is a rational number with a 'not-too-large' denominator. To state our result we introduce some notations: Let $(k, l)=1$, and

$$\Delta(x, l/k)=\sum_{n\leqq x}{}' d(n)\exp(2\pi i l n/k)-\frac{x}{k}\left(\log\frac{x}{k^2}+2\gamma-1\right)-E(0, l/k),$$

where γ is the Euler constant, and $E(0, l/k)$ is the value at $s=0$ of the analytic continuation of

$$E(s, l/k)=\sum_{n=1}^{\infty} d(n)\exp(2\pi i l n/k)n^{-s}.$$

We put

$$\begin{aligned}Y(s, l/k)=&-\exp(\pi i/4)(2\pi/t)^{1/2}(l/k)^{1-s}\Delta(lt/2\pi k, l/k)\\&+\frac{1}{2\sqrt{\pi}}\exp(\pi i/4)(l/k)^{1/2-s}(kl/2\pi t)^{1/4}\sum_{n=1}^{\infty} d(n)\\&\times\exp(-2\pi i\bar{l}n/k)h(n/kl)n^{-3/4},\end{aligned}$$

where $l\bar{l}\equiv 1 \pmod k$ and

$$h(x)=\int_0^{\infty}\exp(-i\pi x\xi)(\xi+1)^{-3/2}d\xi.$$

Theorem. *Let $(k, l)=1$, $l<k$, $kl\leqq t(\log t)^{-20}$. Then we have, for $0\leqq\sigma\leqq 1$,*

$$\chi(1-s)\mathcal{E}_2(s, lt/2\pi k)=Y(s, l/k)+\overline{Y(1-\bar{s}, k/l)}+O((l/k)^{1/2-\sigma}(kl/t)^{1/2}(\log t)^3).$$

Remarks. As has been observed by Jutila ([1, p. 105]), $\mathcal{E}_2(s, \alpha t/2\pi)=\Omega(\log t)$ when α is very close to 1 (e.g. $\alpha=1-ct^{-1/2}$). Thus, if $kl\gg t$ then $\mathcal{E}_2(s, lt/2\pi k)$ cannot be small in general. But our result implies that if kl is relatively small then the approximation becomes significant. This reminds us of the 'major-arc, minor-arc' situation in the theory of trigonometrical method. It should be noted also that the O-term in our theorem may be replaced by an asymptotic series in terms of $(kl/t)^{1/4}$.

2. We show here an outline of the proof. The details will be given elsewhere.

By the splitting argument of Dirichlet we get, as before,

$$\mathcal{E}_2(s, lt/2\pi k)=2\chi(s)k^{s-1}l^{-s}\sum_{n\leqq(t/2\pi kl)^{1/2}} n^{-1}+\{\mathcal{E}_1(s, (lt/2\pi k)^{1/2})\}^2 +2G(s, l/k)+2\chi^2(s)G(1-s, k/l),$$

where

$$G(s, l/k)=\sum_{n\leqq(tl/2\pi k)^{1/2}} n^{-s}\mathcal{E}_1(s, lt/2\pi kn)$$

and

$$\mathcal{E}_1(s, x)=\zeta(s)-\sum_{n\leqq x} n^{-s}-\chi(s)\sum_{n\leqq t/2\pi x} n^{s-1}.$$

We note that the integral representation, due to Riemann and Siegel, of $\mathcal{E}_1(s, x)$ is valid as far as $x\ll t^c$. Thus we have, for $kl\ll t^c$,

$$G(s, l/k)=(2\pi i)^{-1}\chi(s)(l/k)^{1-s}\sum_{n\leqq(tl/2\pi k)^{1/2}} n^{-1}\exp(-2\pi ikn[lt/2\pi kn]/l)$$
$$\times\int_L(\exp(w+2\pi ikn/l)-1)^{-1}\exp(iw^2l^2t/8\pi^2k^2n^2+\{lt/2\pi kn\}w)dw$$
$$+O(\chi(s)(l/k)^{1/2-\sigma}(kl/t)^{1/2}\log t),$$

where $\{x\}=x-[x]$, and L is a straight line in the direction $\arg w=\pi/4$, passing between 0 and $2\pi i$. The transformation of this integral is conducted as in [2], and we see that it is equal to

$$-\delta(kn/l)\pi i+\pi^{1/2}\exp(\pi i/4)(8\pi^2k^2n^2/l^2t)^{1/2}$$
$$\times\left(\left(\{lt/2\pi kn\}-\frac{1}{2}\right)\delta(kn/l)+(\exp(2\pi ikn/l)-1)^{-1}(1-\delta(kn/l))\right)$$
$$+\frac{1}{2}\int_{(5/4)}\Gamma(w)\exp(\pi iw/2)(\cos(\pi w))^{-1}(2k^2n^2/t)^w$$
$$\times\Big(\sum_{m\equiv-kn\,(\mathrm{mod}\ l)} m^{-2w}\exp(2\pi im\{lt/2\pi kn\}/l)$$
$$-\sum_{m\equiv kn\,(\mathrm{mod}\ l)} m^{-2w}\exp(-2\pi im\{lt/2\pi kn\}/l)\Big)dw,$$

where $\delta(x)=1$ if x is an integer, and $=0$ otherwise. Inserting this into the formula for $G(s, l/k)$ we reduce the problem to the asymptotic evaluation of the sums

$$H=\sum_{n\leqq(t/2\pi kl)^{1/2}}\left(\{t/2\pi kn\}-\frac{1}{2}\right)$$
$$+\sum_{\substack{n\not\equiv0\,(\mathrm{mod}\ l)\\ n\leqq(tl/2\pi k)^{1/2}}}(\exp(2\pi ikn/l)-1)^{-1}\exp(-2\pi ikn[lt/2\pi kn]/l)$$

and

$$K(w, m)=\sum_{\substack{n\equiv-\bar{k}m\,(\mathrm{mod}\ l)\\ n\leqq(lt/2\pi k)^{1/2}}} n^{2w-1}\exp(imt/kn)-\sum_{\substack{n\equiv\bar{k}m\,(\mathrm{mod}\ l)\\ n\leqq(lt/2\pi k)^{1/2}}} n^{2w-1}\exp(-imt/kn)$$
$$=2il^{-1}\sum_{f=0}^{l-1}\exp(2\pi if\bar{k}m/l)\sum_{n\leqq(lt/2\pi k)^{1/2}} n^{2w-1}\sin(mt/kn+2\pi fn/l).$$

In fact we have

$$G(s, l/k)=-\frac{1}{2}\chi(s)k^{s-1}l^{-s}\sum_{n\leqq(t/2\pi kl)^{1/2}} n^{-1}+\exp(-\pi i/4)(2\pi/t)^{1/2}(k/l)^sH$$
$$+M+O(\chi(s)(l/k)^{1/2-\sigma}(kl/t)^{1/2}\log t),$$

where

$$M=(4\pi i)^{-1}\chi(s)(l/k)^{1-s}$$
$$\times\int_{(5/4)}\Gamma(w)\exp(\pi iw/2)(\cos(\pi w))^{-1}(2k^2/t)^w\sum_{m=1}^{\infty}m^{-2w}K(w,m)dw.$$

By an elementary computation one may conclude that

$$H=-\frac{1}{2}\Delta(lt/2\pi k,\ -k/l)-\frac{1}{4}d(lt/2\pi k)\exp(-it)+O(l\log(2l)).$$

To $K(w,m)$ we apply the summation formula of Poisson, and evaluate resulting integrals by the saddle point method. The asymptotic result thus obtained is inserted into the w-integral in the formula for M, and we find that M is equal to

$$-\frac{1}{2}\exp(-\pi i/4)(l/k)^{1/2-s}(kl/2\pi t)^{1/4}\sum_{n=1}^{\infty}d(n)n^{-3/4}\exp(2\pi i\bar{k}n/l)h(-n/kl)$$
$$+O(\chi(s)(l/k)^{1/2-\sigma}(kl/t)^{1/2}(\log t)^3).$$

Collecting these we end the proof.

References

[1] A. Ivić: The Riemann zeta-function. John Wiley and Sons Inc., New York (1985).

[2] Y. Motohashi: A note on the approximate functional equation for $\zeta^2(s)$. II. Proc. Japan Acad., **59**A, 469–472 (1983).