114. A Note on the Mean Value of the Zeta and L-functions. V

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1. In the previous note of this series we showed an alternative approach to Atkinson's formula. Here we return to the original argument of Atkinson [1], and exploit its ability in the context of the problem dealt by Balasubramanian, Conrey and Heath-Brown [2]. Motivated by Iwaniec [3], they considered the asymptotic evaluation of

$$I(T, A)=\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)A\left(\frac{1}{2}+it\right)\right|^2 dt,$$

where

$$A(s)=\sum_m a(m)m^{-s}$$

and $a(m)$ vanishes for $m>M$. The main term of the integral is

$$T\sum_{k,l}\frac{a(k)\bar{a}(l)}{[k, l]}\left(\log\frac{T(k, l)^2}{2\pi kl}+2\gamma-1\right),$$

and denoting the error-term by $E(T, A)$, they proved, among other things, that

$$E(T, A)\ll T(\log T)^{-B}+M^2T^{\varepsilon}$$

for any fixed B, $\varepsilon>0$ whenever $\log M\ll\log T$, $a(m)\ll m^{\varepsilon}$. Thus $I(T, A)$ is asymptotically equal to the main-term when $M<T^{(1/2)-\varepsilon}$.

Their argument is highly technical, and centers upon a subtle estimation of integrals arising from a Mellin transform of the Γ-factor in the functional equation for $\zeta(s)$. In contrast with this, as we shall show below, a simple modification of Atkinson's argument yields a quite accessible proof of the above as well as the following new estimate:

Theorem.

$$E(T, A)\ll T^{1/3}M^{4/3}T^{\varepsilon}.$$

Remark. (i) Assertions (B) and (C) stated in [2, Theorem 1] can also be proved by refining our argument.

(ii) Our result contains $E(T)\ll T^{1/3+\varepsilon}$.

(iii) The mean square of $E(T, A)$ may be considered. And we stress that in application to the problem of the distribution of the zeros of $\zeta(s)$ as was done in [2] a good mean value estimate of $E(T, A)$ is enough.

(iv) The χ-analogue of our result can be obtained by combining the present note with [4, II].

2. Now we shall show an outline of our argument. The details will be given elsewhere.

We have, for $Re(u)>1$, $Re(v)>1$,

$$\zeta(u)\zeta(v)A(u)\overline{A(\bar{v})}=\zeta(u+v)\sum_{k,l}a(k)\bar{a}(l)[k,l]^{-u-v}+M(u,v)+\overline{M(\bar{v},\bar{u})},$$

where

$$M(u,v)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Big(\sum_{k|m}a(k)\Big)\Big(\sum_{l|m+n}\bar{a}(l)\Big)m^{-u}(m+n)^{-v}.$$

An analytic continuation of $M(u,v)$ to the region $Re(u)<1$ may be obtained by following the argument of [4, II]; we have

$$M(u,v)=\Gamma(u+v-1)\Gamma(1-u)\Gamma(v)^{-1}\zeta(u+v-1)\sum_{k,l}\frac{(k,l)^{1-u-v}}{[k,l]}a(k)\bar{a}(l)$$
$$+g(u,v;A),$$

where

$$g(u,v;A)=\{\Gamma(u)\Gamma(v)(e^{2\pi iu}-1)(e^{2\pi iv}-1)\}^{-1}\sum_{k,l}a(k)\bar{a}(l)l^{-1}$$
$$\times\sum_{f=1}^{l}\int_C y^{v-1}\Big(\exp\Big(y-2\pi i\frac{f}{l}\Big)-1\Big)^{-1}$$
$$\times\int_C x^{u-1}\Big(\Big(\exp\Big(k(x+y)-2\pi i\frac{fk}{l}\Big)-1\Big)^{-1}-\frac{\delta(f)}{k(x+y)}\Big)dxdy.$$

Here $\delta(f)=1$ if $l\,|\,kf$, and $=0$ if $l\nmid kf$, and C is as in [4, I]. Collecting these and letting $u+v$ tend to 1, we have, for $0<Re(u)<1$,

$$\zeta(u)\zeta(1-u)A(u)\overline{A(1-\bar{u})}$$
$$=\sum_{k,l}\frac{a(k)\bar{a}(l)}{[k,l]}\Big\{\frac{1}{2}\Big(\frac{\Gamma'}{\Gamma}(u)+\frac{\Gamma'}{\Gamma}(1-u)\Big)+\log\frac{(k,l)^2}{kl}+2\gamma-\log 2\pi\Big\}$$
$$+g(u,1-u;A)+g(1-u,u;\bar{A}).$$

Again as in [4, I] we have, for $Re(u)<0$,

$$g(u,1-u;A)=\sum_{k,l}\frac{a(k)\bar{a}(l)}{[k,l]}\sum_{n\neq 0}d(|n|)\exp\Big(2\pi i\frac{\bar{k}^*}{l^*}n\Big)$$
$$\times\int_0^{\infty}\exp\Big(2\pi i\frac{ny}{k^*l^*}\Big)y^{-u}(y+1)^{u-1}dy,$$

where $k/(k,l)=k^*$, $l/(k,l)=l^*$, and $k^*\bar{k}^*\equiv 1 \bmod l^*$. Then we reach an expression for $I(T,A)$ which corresponds precisely to [1, (4.4)]. We take an exponential-average of this as was done in [4, II] and find eventually that, for any $G\leq T(\log T)^{-1}$,

$$E(T,A)\ll(G+M)T^{\varepsilon}+\sum_{k,l}\frac{|a(k)a(l)|}{[k,l]}\operatorname*{Max}_{T/2<V<2T}(|P_1|+|P_2|+|P_3|),$$

where

$$P_1=\sum_{n\leq N}d(n)\exp\Big(2\pi i\frac{\bar{k}^*}{l^*}n\Big)\int_0^{\infty}\exp\Big(2\pi i\frac{n}{l^*k^*}y\Big)\frac{\sin(V\log(1+1/y))}{(y(y+1))^{1/2}\log(1+1/y)}$$
$$\times\exp\Big(-\frac{1}{4}(G\log(1+1/y))^2\Big)dy,$$

$$P_2=\Delta\Big(N+\frac{1}{2},\frac{\bar{k}^*}{l^*}\Big)\int_0^{\infty}\exp\Big(2\pi i\frac{(N+1/2)y}{k^*l^*}\Big)\frac{\sin(V\log(1+1/y))}{(y(y+1))^{1/2}\log(1+1/y)}$$
$$\times\exp\Big(-\frac{1}{4}(G\log(1+1/y))^2\Big)dy,$$

$$P_3=\int_{N+1/2}^{\infty} x^{-1}\Delta\left(x, \frac{\bar{k}^*}{l^*}\right)\int_0^{\infty}\frac{\exp(2\pi i(xy/k^*l^*))}{y^{1/2}(1+y)^{3/2}\log(1+1/y)}\Big\{V\cos(V\log(1+1/y))$$
$$-\left(\frac{1}{2}+\frac{1}{2}G^2\log(1+1/y)+(\log(1+1/y))^{-1}\right)\sin(V\log(1+1/y))\Big\}$$
$$\times\exp\left(-\frac{1}{4}(G\log(1+1/y))^2\right)dxdy.$$

Here

$$\Delta\left(x, \frac{\bar{k}^*}{l^*}\right)=\sum_{n\leqq x} d(n)\exp\left(2\pi i\frac{\bar{k}^*}{l^*}n\right)-\frac{x}{l^*}(\log x+2\gamma-1-2\log l^*)-D\left(0, \frac{\bar{k}^*}{l^*}\right);$$

$$D\left(s, \frac{\bar{k}^*}{l^*}\right)=\sum_{n=1}^{\infty} d(n)\exp\left(2\pi i\frac{\bar{k}^*}{l^*}n\right)n^{-s}.$$

And the integer $N\approx k^*l^*T$ is to satisfy

$$\Delta\left(N+\frac{1}{2}, \frac{\bar{k}^*}{l^*}\right)\ll l^{*1/2}N^{1/4}+l^*T^{\varepsilon}.$$

This is possible, for we have

$$\int_X^{2X}\left|\Delta\left(x, \frac{\bar{k}^*}{l^*}\right)\right|^2 dx\ll l^*X^{3/2}+l^*X^{1+\varepsilon},$$

which is a consequence of the analogue for $\Delta\left(x, \frac{\bar{k}^*}{l^*}\right)$ of the classical truncated Voronoi formula for $\Delta(x)$. The estimation of P_1, P_2, P_3 is made in much the same way as in [4, II]. And we obtain

$$E(T, A)\ll(G+(T/G)^{1/2}M^2)T^{\varepsilon}$$

which obviously gives rise to our theorem.

References

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