109. *Isotropic Submanifolds in a Euclidean Space*

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(Communicated by Kôsaku YOSIDA, M. J. A., Dec. 12, 1986)

The Gauss map of a submanifold M in a Euclidean n-space E^n is the map which is obtained by the parallel displacement of the tangent plane of M in E^n. It is well known that the image of an m-dimensional submanifold in E^n by the Gauss map lies in the Grassmann manifold $G(m, n-m)$. The Gauss map is useful for the study of submanifolds in E^n.

In the present paper we will discuss isotropic submanifolds in E^n with conformal Gauss map and prove the following

Theorem. *Let M be an m-dimensional Riemannian manifold isotropically immersed in E^n. If the Gauss map Γ is conformal and the image $\Gamma(M)$ is totally umbilical in $G(m, n-m)$, then M is a minimal and isotropic submanifold in a hypersphere S^{n-1} of E^n with the parallel second fundamental form.*

We well know that minimal isotropic submanifolds in a sphere with the parallel second fundamental form are classified in [5].

§ 1. **Preliminaries.** In the present paper we use the notations introduced in [3] and [4]. Let M be an m-dimensional Riemannian manifold immersed in E^n through the isometric immersion ι. In each neighborhood $V \subset M$, M is given by differentiable functions

(1.1) $$x^A = x^A(y^1, y^2, \cdots, y^m),$$

where x^A $(A=1, 2, \cdots, n)$ are rectangular coordinates of E^n and y^i $(i=1, 2, \cdots, m)$ local coordinates of M in V. We define B_i^A by $B_i^A = \partial x^A/\partial y^i$. The tangent plane $\iota(M_p)$, $p \in M$, of ιM may be considered as a point $\Gamma(p)$ of $G(m, n-m)$ by the parallel displacement in E^n, and so we get naturally a mapping $\Gamma: M \to G(m, n-m)$ which is called *the Gauss map* associated with the immersion ι and $\Gamma(M)$ *the Gauss image* of M. In the present paper, we always assume that *the Gauss map is regular.*

Now, we assume that $V \subset M$ is a neighborhood of a fixed point $p \in M$ whose local coordinates satisfy $y^i = 0$, $i=1, \cdots, m$. Let (e_i, e_α) be a fixed orthonormal frame of E^n such that e_i are vectors of $\iota(M_p)$ and e_α are normal to $\iota(M_p)$. For each point $q \in V$, let (f_i, f_α) be an orthonormal frame of E^n where f_i are vectors of $\iota(M_q)$ and f_α are normal to $\iota(M_q)$ such that, in V, (f_i, f_α) is a differentiable frame satisfying $\langle f_i, e_j\rangle = \langle f_j, e_i\rangle$, $\langle f_\alpha, e_\beta\rangle = \langle f_\beta, e_\alpha\rangle$ and $f_i(0) = e_i$, $f_\alpha(0) = e_\beta$. Denoting f_i^A the components of the vector f_i, we may put $f_i^A = \sum_k \gamma_i^k B_k^A$. The matrix (γ_j^i) satisfies $\sum \gamma_i^l \gamma_j^k g_{lk} = \delta_{ij}$, $g_{ij} = \sum B_i^A B_j^A$, where g_{ij} are the components of the first fundamental form g of M. Then we have $\sum \gamma_l^i \gamma_l^j = g^{ij}$ where $\sum g^{ik} g_{kj} = \delta_j^i$. The components of the second fundamental form are

(1.2) $$h^A_{ij}=\nabla_i B^A_j=\partial_i B^A_j-\sum \{{}_i{}^k{}_j\}B^A_k,$$

where $\{{}_i{}^k{}_j\}$ are the Christoffel symbols derived from g_{ij} and ∇ is the covariant differentiation with respect to the metric g. For each point $q\in V$, the image $\Gamma(q)$ is the m-plane spanned by $f_1, \cdots, f_m$. The distance $d\sigma$ between two points $\Gamma(q)$ and $\Gamma(q+dq)$ is given by

$$(d\sigma)^2=\sum \langle df_i, f_\alpha\rangle^2=\sum g^{ij}h^A_{il}h^A_{jk}dy^l dy^k,$$

where dy^i are differences between the local coordinates of the points $q+dq$ and q. From this formula, we see that the Riemannian metric G of $\Gamma(M)$ is given by

(1.3) $$G_{ij}=\sum g^{kl}h^A_{ik}h^A_{jl}.$$

Since Γ is assumed to be regular, M admits two metric g and G, one induced from ι and the other induced from its Gauss map Γ. The Gauss map is said to be *conformal* if $G=e^{2\rho}g$ for some differentiable function ρ on M. If the above function ρ is constant on M, then the Gauss map is said to be *homothetic*.

Let Π be a point of $G(m, n-m)$. As stated in [4], we choose a system of local coordinates $(\xi_{i\alpha})$ in a suitable open neighborhood U of Π, where the indices run as follows: $i=1, 2, \cdots, m$; $\alpha=m+1, \cdots, n$. Then, the components of the second fundamental form of $(\Gamma(M), G)$ in $(G(m, n-m), \tilde{g})$ are given by, in $\Gamma(V)\subset U$,

$$\tilde{h}^{i\alpha}_{jk}=\partial^2\xi_{i\alpha}/\partial y^j\partial y^k-{}^G\{{}_j{}^l{}_k\}\partial\xi_{i\alpha}/\partial y^l+\{{}_{\iota\beta}{}^{i\alpha}{}_{h\gamma}\}(\partial\xi_{\iota\beta}/\partial y^j)(\partial\xi_{h\gamma}/\partial y^k),$$

where ${}^G\{{}_j{}^l{}_k\}$ is the Christoffel symbols of $(\Gamma(M), G)$ and $\{{}_{\iota\beta}{}^{i\alpha}{}_{h\gamma}\}$ is the ones of $(G(m, n-m), \tilde{g})$.

The immersion is said to be *λ-isotropic* if the second fundamental form satisfies (see [1]),

(1.4) $$\sum h^A_{ij}h^A_{kl}+\sum h^A_{ik}h^A_{jl}+\sum h^A_{il}h^A_{jk}=\lambda^2(g_{ij}g_{kl}+g_{ik}g_{jl}+g_{il}g_{jk}),$$

where λ is a differentiable function on M.

§2. The proof of Theorem. At first, we will prove the following

Proposition 1. *Let M be an m-dimensional Riemannian submanifold which is λ-isotropically immersed in E^n. If $m\geqq 3$ and Gauss map Γ is conformal, then Γ is homothetic.*

Proof. From (1.4), we have

(2.1) $$m\sum h^A h^A_{ij}+2\sum g^{kl}h^A_{ik}h^A_{lj}=(m+2)\lambda^2 g_{ij}, \qquad \text{where } h^A=\frac{1}{m}\sum g^{kl}h^A_{kl}.$$

On the other hand, the Ricci curvature K_{ij} is given by

(2.2) $$K_{ij}=m\sum h^A h^A_{ij}-\sum h^A_{ik}h^A_{lj}g^{kl}.$$

It follows from (2.1) and (2.2) that we have

(2.3) $$2K_{ij}=3m\sum h^A h^A_{ij}-(m+2)\lambda^2 g_{ij},$$

(2.4) $$\sum h^A h^A_{ij}=\frac{1}{3m}\{(m+2)\lambda^2 g_{ij}+2K_{ij}\},$$

(2.5) $$G_{ij}=\sum g^{kl}h^A_{ik}h^A_{lj}=\frac{1}{3}\{(m+2)\lambda^2 g_{ij}-K_{ij}\},$$

which imply the following

Lemma 1. *The Gauss map Γ is conformal if and only if M is Einsteinian or it is pseudo-umbilical.*

Then, since $m \geqq 3$, the above results imply that M is Einsteinian and

$$(2.6) \qquad G_{ij} = \frac{1}{3}\left\{(m+2)\lambda^2 - \frac{K}{m}\right\} g_{ij}, \qquad K = \text{constant},$$

where K is the scalar curvature of M.

Now, let $G_{ij} = \rho g_{ij}$, where ρ is a differentiable function on M. Since $G_{ij} = \sum g^{kl} h^A_{ik} h^A_{jl}$, we have

$$\sum g^{kl} (\nabla_r h^A_{jk}) h^A_{il} + \sum g^{kl} (\nabla_r h^A_{il}) h^A_{jk} = \rho_r g_{ij},$$

which implies

$$(2.7) \qquad \sum g^{kl} (\nabla_k h^A) h^A_{il} + \sum g^{kl} (\nabla_l h^A_{ir}) h^A_{jk} g^{jr} = \rho_i,$$

$$(2.8) \qquad \sum g^{ij} g^{kl} (\nabla_r h^A_{jk}) h^A_{il} = \frac{1}{2} \rho_r g^{ij} g_{ij} = (m/2) \rho_r.$$

It follows from (2.7) and (2.8) that

$$(2.9) \qquad \sum g^{kl} (\nabla_k h^A) h^A_{il} = -((m-2)/2) \rho_i.$$

On the other hand, from (1.4) we have

$$(2.10) \qquad \sum (\nabla_l h^A) h^A_{ij} + 2\nabla_l G_{ij} = c_l g_{ij}, \qquad \text{where } c = (m+2)\lambda^2.$$

Since K is constant, (2.6) implies $\nabla_l G_{ij} = (c_l/3) g_{ij}$. Then from (2.10) we have

$$\sum (\nabla_l h^A) h^A_{ij} = (c_l/3) g_{ij} = \rho_l g_{ij}, \qquad \text{because of } \rho = \frac{1}{3}\left(c - \frac{K}{m}\right),$$

which implies

$$(2.11) \qquad \sum g^{lj} (\nabla_l h^A) h^A_{ij} = \rho_i.$$

It follows from (2.9) and (2.11) that

$$m\rho_i = 0, \qquad \text{that is, } \rho \text{ must be constant on } M.$$

Thus the Gauss map Γ is homothetic. Q.E.D.

Since the local expression $\xi_{i\alpha} = \xi_{i\alpha}(y)$ of the immersion from M into $G(m, n-m)$ is given by $\xi_{i\alpha} = \langle f_i, e_\alpha \rangle = \sum \gamma^j_i B^A_j e^A_\alpha$, we have

$$(2.12) \qquad \tilde{B}^{j\alpha}_i = (\partial \xi_{j\alpha})/(\partial y^i) = \sum \{(\nabla_i \gamma^k_j) B^A_k + \gamma^k_j h^A_{ik}\} e^A_\alpha.$$

Since Γ is homothetic, $\{{}_i{}^l{}_j\} = {}^G\{{}_i{}^l{}_j\}$. As stated in [4], we have

$$(2.13) \qquad \tilde{h}^{i\alpha}_{jk} = \nabla_j \tilde{B}^{i\alpha}_k = \sum \gamma^l_i \sum (\nabla_j h^A_{kl}) e^A_\alpha \qquad \text{at } p \in V,\ y^i(p) = 0.$$

Now, we must state the following Muto's Theorem 3.5 in [4].

Lemma 2. *If the Gauss map Γ is homothetic and the Gauss image $\Gamma(M)$ is totally umbilical in $G(m, n-m)$, then $\Gamma(M)$ is totally geodesic.*

By this Lemma 2 and (2.13) we have

$$\sum N^A (\nabla_i h^A_{jk}) = 0, \qquad \text{for every normal vector } N \text{ to } M_p,$$

that is, the second fundamental form of M is parallel in the normal bundle. Thus, we have proved the following

Proposition 2. *Let M be an m-dimensional Riemannian manifold isotropically immersed in E^n, $m \geqq 3$. If the Gauss map Γ is conformal and the Gauss image $\Gamma(M)$ is totally umbilical in $G(m, n-m)$, then M is Einsteinian, the Gauss map Γ is homothetic and the second fundamental form is parallel in the normal bundle.*

By this Proposition and Theorem 4.2 in [4], we see that *M is a minimal and isotropic submanifold in a hypersphere* S^{n-1} *of* E^n. In this case, we easily see that the second fundamental form of M in S^{n-1} is parallel in the normal bundle. Therefore, we have proved our main Theorem.

References

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