98. On Closed Maximal Ideals of M*),†)

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(Communicated by Kôsaku YOSIDA, M. J. A., Nov. 12, 1986)

1. Introduction. Let U be the unit disc $\{|z|<1\}$ in $\boldsymbol{C}$. A function f holomorphic in U is said to belong to the class M if

$$\int_0^{2\pi} \log^+ Mf(\theta)\frac{d\theta}{2\pi}<\infty,$$

where $Mf(\theta)=\sup_{0\le r<1}|f(re^{i\theta})|$ and $\log^+ x=\max(\log x, 0)$, $x>0$. The class M was introduced and studied in [3]. It is shown that

$$\bigcup_{p>0} H^p \subsetneqq M \subsetneqq N^+,$$

where H^p is the usual Hardy class of order $p>0$ and N^+ the Smirnov class. See [1] or [2] for the general theory of H^p and N^+.

The space M with the metric given by

$$d(f, g)=\int_0^{2\pi} \log(1+M(f-g)(\theta))\frac{d\theta}{2\pi}$$

is an F-algebra, i.e., a topological vector space with a complete translation invariant metric in which multiplication is continuous. The class M has many similarities with the Smirnov class N^+ as an F-algebra. See [3] and [4]. For example, the following are noted in [3].

(1) For $\lambda\in U$, if we define

$$\gamma_\lambda(f)=f(\lambda), \qquad f\in M,$$

then γ_λ is a continuous multiplicative linear functional on M. Conversely, if γ is a nontrivial multiplicative linear functional on M then $\gamma=\gamma_\lambda$ for some $\lambda\in U$.

(2) If $\lambda\in U$ and $m_\lambda=\{f\in M: f(\lambda)=0\}$ then $m_\lambda=(z-\lambda)M$ and m_λ is a closed maximal ideal of M.

(3) There exists a maximal ideal m of M which is not the kernel of a multiplicative linear functional on M.

In this note, we show that every closed maximal ideal is the kernel of a multiplicative linear functional on M (see Corollary 5). The corresponding theorem for N^+ was proved [4].

2. Main theorem.

Lemma 1. *Let m be a nonzero ideal of M. Then m contains a bounded holomorphic function which is not identically zero.*

Proof. Let $f\in m$ and $f\not\equiv 0$. Since $M\subset N^+$, f can be factored canoni-

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*) Dedicated to Professor Mun-Gu Sohn on his 60th birthday.

†) This research was partly supported by KOSEF (1986).

cally as follows [1]:

$$f(z)=B(z)S(z)F(z),$$

where B is the Blaschke product with respect to the zeros of f, S the singular inner function associated with f, and

$$F(z)=\exp\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log|f(e^{it})|\,dt\right),$$

the outer function associated with f. If we set

$$g(z)=\exp\left(-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log^+|f(e^{it})|\,dt\right),$$

then $g\in M$. In fact, g is bounded. Since m is an ideal of M, $fg\in m$ and

$$f(z)g(z)=B(z)S(z)\exp\left(-\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}\log^-|f(e^{it})|\,dt\right)$$

is bounded. This completes the proof.

Lemma 2. *Suppose that $F\in M$ never vanishes on U. Then there exists a sequence $\{F_n\}$ of functions F_n in M such that $F_n{}^n=F$ and $F_n\to 1$ in M as $n\to\infty$.*

Proof. Since F never vanishes on U, there exists a positive continuous function $\Theta(z)$ on U such that

$$F(z)=R(z)e^{i\Theta(z)},\qquad z\in U.$$

We define

$$F_n(z)=R(z)^{1/n}e^{i(1/n)\Theta(z)},\qquad z\in U.$$

Then $F_n(z)$ is holomorphic in U and $F_n{}^n=F$. We note that F as a nonzero function of N^+ has nonzero radial limits almost every θ. We fix such a θ. Then $R(re^{i\theta})$ is a positive continuous function of r on the closed interval $[0,1]$; so we can find positive numbers l_θ and L_θ so that

$$0<l_\theta\le R(re^{i\theta})\le L_\theta<\infty,\qquad 0\le r\le 1.$$

$\Theta(re^{i\theta})$ also being a continuous function of r on $[0,1]$ it is bounded. Therefore we can conclude that

$$F_n(re^{i\theta})\to 1,\qquad (n\to\infty),$$

uniformly on r $(0\le r\le 1)$. Hence we have

$$M(F_n-1)(\theta)\to 0,\quad (n\to\infty),\qquad \text{a.e. } \theta.$$

We note that

$$\log^+ MF_n(\theta)\le\frac{1}{n}\log^+ MF(\theta)\le\log^+ MF(\theta),\qquad n=1,2,\cdots,$$

and

$$\begin{aligned}\log(1+M(F_n-1)(\theta))&\le\log 2+\log(MF_n(\theta)+1)\\&\le 2\log 2+\log^+ MF_n(\theta)\\&\le 2\log 2+\log^+ MF(\theta),\qquad n=1,2,\cdots.\end{aligned}$$

We have $F_n\in M$ and $d(F_n,1)\to 0$ as $n\to\infty$ by the dominated convergence theorem. This completes the proof.

Lemma 3. *Let B be an infinite Blaschke product and let $B(z)=B_n(z)g_n(z)$, where*

$$B_n(z)=\prod_{k=1}^{n}\frac{|a_k|}{a_k}\,\frac{a_k-z}{1-\bar a_k z},$$

and

$$g_n(z)=\prod_{n+1}^{\infty}\frac{|a_k|}{a_k}\frac{a_k-z}{1-\bar{a}_k z}.$$

Then $g_n \to 1$ *in* M *as* $n\to\infty$.

Proof. If we note that $\log(1+x)\le x$ $(x>0)$ and use Hölder's inequality, we have

$$\begin{aligned} d(g_n,1)&=\int_0^{2\pi}\log(1+M(g_n-1)(\theta))\frac{d\theta}{2\pi}\\ &\le\int_0^{2\pi}M(g_n-1)(\theta)\frac{d\theta}{2\pi}\\ &\le\left(\int_0^{2\pi}M(g_n-1)(\theta)^2\frac{d\theta}{2\pi}\right)^{1/2}. \end{aligned}$$

We now apply the complex maximal theorem and use the fact that $|B_n(e^{i\theta})|=1$ to get

$$\begin{aligned} d(g_n,1)&\le C\left(\int_0^{2\pi}|g_n(e^{i\theta})-1|^2\frac{d\theta}{2\pi}\right)^{1/2}\\ &=C\left(\int_0^{2\pi}|B(e^{i\theta})-B_n(e^{i\theta})|^2\frac{d\theta}{2\pi}\right)^{1/2} \end{aligned}$$

where C is a positive constant. By [2, p. 66], the last term in the above inequality tends to zero as $n\to\infty$. Therefore $g_n\to 1$ in M.

Theorem 4. *Let* m *be a nonzero prime ideal of* M *which is not dense in* M. *Then* $m=m_\lambda$ *for some* $\lambda\in U$.

Proof. Suppose that $m\neq m_\lambda$ for any $\lambda\in U$. By Lemma 1, m contains a bounded holomorphic function f. We know that f can be factored as $f=BF$ where B is the Blaschke product with respect to the zeros of f and F is a bounded function with no zeros. Since m is prime, either $F\in m$ or $B\in m$. Suppose $F\in m$ and let F_n be defined as in Lemma 2. Then $F_n \in m$ by the primeness of m. By Lemma 2, $1\in\overline{m}$; so $\overline{m}=M$, a contradiction. Now, we suppose $B\in m$ and let

$$B(z)=\prod_k\frac{|a_k|}{a_k}\frac{a_k-z}{1-\bar{a}_k z}.$$

If $(a_k-z)/(1-\bar{a}_k z)\in m$, then

$$m_{a_k}=\frac{a_k-z}{1-\bar{a}_k z}M\subset m;$$

so $m=m_{a_k}$ by the maximality of m_{a_k}, a contradiction. Therefore

$$\frac{a_k-z}{1-\bar{a}_k z}\notin m,\qquad k=1,2,\cdots.$$

Since m is prime, B should be an infinite Blaschke product. If we set

$$g_n(z)=\prod_{k\ge n}\frac{|a_k|}{a_k}\frac{a_k-z}{1-\bar{a}_k z},\qquad n=1,2,\cdots,$$

then $g_n\in m$ by the primeness of m. By Lemma 3, $g_n\to 1$ in M as $n\to\infty$. Therefore $1\in m$, a contradiction. Hence we conclude that $m=m_\lambda$ for some $\lambda\in U$. This completes the proof.

Corollary 5. *Every closed maximal ideal of M is the kernel of a multiplicative linear functional.*

Proof follows from Theorem 4 and (1).

Acknowledgements. This research was completed during the author's visit to Chiba University. The author would like to express his thanks to the members of mathematics department for the hospitality and especially to Professor N. Yanagihara for several helpful conversations. The author also would like to express his thanks to JSPS and KOSEF for the financial support.

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