8. Special Values of Euler Products and Hardy-Littlewood Constants

By Nobushige KUROKAWA

Department of Mathematics, Tokyo Institute of Technology

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We note an interpretation of Hardy-Littlewood constants (originally constructed by Hardy-Littlewood [2] and generally by Bateman-Horn [1]) as special values of certain Euler products treated in author's papers [3], [4], [5] supplementing the ending remark of [6]. The contents of this report were presented in a seminar of Research Center for Advanced Mathematics on "spatial zeta functions" at Nagoya University in November 1985. The author is grateful to thank Professor T. Sunada for his invitation to that seminar with supplying important information containing vast results of his school partially summarized in Sunada [9], and would like to express hearty thanks to Professor K. Shiga for making that opportunity.

Let $f(X) \in Z[X]$ be a separable primitive polynomial in one variable with coefficients in the rational integers Z. For each prime number p we put $N(p, f) = \#\{x \in F_p; \bar{f}(x) = 0\}$, where $\bar{f}(X) \in F_p[X]$ denotes the reduction of $f(X)$ modulo p, F_p being the finite field of p elements, and $\#$ denotes the cardinality, so $0 \leq N(p, f) \leq p$. We define the "zeta function" $Z(s, f)$ of f by

$$Z(s, f) = \prod_p (1 - N(p, f)p^{-s})$$

where p runs over all prime numbers. (We do not take the inverse since the above form is suitable for our purpose below.) Let $M(f) = \mathrm{Spec}\,(Z[X]/(f))$ be the scheme associated with f over $\mathrm{Spec}\,(Z)$, then the Hasse-Weil zeta function $\zeta(s, M(f))$ of the one dimensional space $M(f)$ is equal to

$$\zeta(s, M(f)) = \prod_p (1 - N(p, f)p^{-s} + \cdots)^{-1},$$

so we can consider $Z(s, f)$ as a truncated zeta function of $M(f)$. Now the Hardy-Littlewood constant $C(f)$ of $f(X)$ is defined via

$$C(f) = \prod_p (1 - N(p, f)p^{-1})(1 - p^{-1})^{-r(f)}$$

where $r(f)$ denotes the number of irreducible factors of $f(X)$ in $Z[X]$, and p runs over all prime numbers according to the natural order $2, 3, \cdots$ (since this infinite product does not converge absolutely in general).

Theorem A1. *Let $f(X)$ be as above.*

(1) *$Z(s, f)$ is meromorphic in $\mathrm{Re}\,(s) > 0$. It is meromorphic on C if $\deg(f) \leq 1$, and otherwise it has the natural boundary $\mathrm{Re}\,(s) = 0$.*

(2) *$Z(s, f)$ is holomorphic in $\mathrm{Re}\,(s) \geq 1$ and has the following Taylor expansion at $s = 1$:*

$$Z(s, f) = C(f)(s-1)^{r(f)} + (\text{higher order terms}).$$

Proof. We denote by $D(f)$ the discriminant of $f(X)$. Then $D(f)$ is a non-zero rational integer. Let $f(X) = f_1(X) \cdots f_r(X)$ be the decomposition into irreducible factors with $r = r(f)$; $f_i(X)$ are separable primitive irreducible primitive polynomials in $Z[X]$. Let K be the splitting field of $f(X)$ over the rational number field Q and put $K_i = Q[X]/(f_i)$ considering K_i as a subfield of K. We put $G = \mathrm{Gal}(K/Q)$ and $H_i = \mathrm{Gal}(K/K_i)$. Let $\sigma_i = \mathrm{Ind}_{H_i}^G(1)$ be the permutation representation of G on G/H_i, and put $\sigma = \sigma_1 \oplus \cdots \oplus \sigma_r$. Let $\zeta(s, K_i) = \sum_{n=1}^{\infty} c_i(n) n^{-s}$ be the expansion of the Dedekind zeta function of K_i. We denote by P the set of all prime numbers not dividing $D(f)$. We define an Euler datum $E = (P, G, \alpha)$ via $G = \mathrm{Gal}(K/Q)$ and the Frobenius map $\alpha : P \to \mathrm{Conj}(G)$ (remark that each $p \in P$ is unramified in K/Q, so α in unique). Let $p \in P$ then

$$N(p, f) = \sum_{i=1}^{r} N(p, f_i) = \sum_{i=1}^{r} c_i(p) = \sum_{i=1}^{r} \mathrm{tr}\,(\sigma_i(\alpha(p))) = \mathrm{tr}\,(\sigma(\alpha(p))).$$

(Remark that $c_i(p)$ is equal to the number of prime ideals of K_i over p of degree 1.) Hence putting $H(T) = 1 - \mathrm{tr}\,(\sigma)T$ we have

$$Z(s, f) = L(s, E, H)^{-1} \prod_{p | D(f)} (1 - N(p, f)p^{-s}).$$

Since $r(H) = \deg(f)$, the part (1) follows from [3, Theorem 1] (see [4-I, Theorem 1] and [5-I, Theorem 1] also). For the part (2) note that the multiplicity of the trivial representation in σ is r, hence $\zeta(s)^r Z(s, f)$ is holomorphic at $s = 1$ and

$$\lim_{s \to 1} (s-1)^{-r} Z(s, f) = \lim_{s \to 1} \zeta(s)^r Z(s, f) = C(f). \qquad \text{Q.E.D.}$$

Remark A1. It occurs that $C(f) = 0$ when $N(p, f) = p$ for some p (an example : $f(X) = X(X+1)$, $N(2, f) = 2$).

Remark A2. It is natural that $Z(s, f)$ would be holomorphic in $\mathrm{Re}\,(s) > 1/2$.

Let $f(X)$, $E = (P, G, \alpha)$ and $\sigma : G \to \mathrm{GL}(\deg(f), C)$ be as above, and let

$$H(T) = 1 - \mathrm{tr}\,(\sigma)T = \prod_{n=1}^{\infty} \prod_{\rho} \det(1 - \rho T^n)^{\kappa(n, \rho)}$$

be the canonical expansion as in [4-I] and [5-II], where ρ runs over irreducible representations of G and $\kappa(n, \rho)$ are inductively constructed rational integers; in this special case we moreover have an explicit formula for $\kappa(n, \rho)$ which shows that $\kappa(n, \rho)$ are *non-negative* and

$$\sum_{\rho} \kappa(n, \rho) \deg(\rho) = n^{-1} \sum_{m | n} \mu(n/m)(\deg(f))^m.$$

Let $p(f)$ be the maximal prime divisor of $D(f)$.

Theorem A2. *For each integer $M \geq \max(p(f), \deg(f))$ we have*

$$C(f) = \prod_{p \leq M} (1 - N(p, f)p^{-1})(1 - p^{-1})^{-r(f)} \prod_{(n, \rho) \neq (1,1)} L^M(n, E, \rho)^{-\kappa(n, \rho)}$$

where

$$L^M(s, E, \rho) = \prod_{p > M} \det(1 - \rho(\alpha(p))p^{-s})^{-1}.$$

Proof. This follows from Theorem A1 (2) with trivial estimations showing that the above product converges absolutely (and moreover rapidly).   Q.E.D.

For $f(X)=f_1(X)\cdots f_r(X)$ and K_i as in the proof of Theorem A1 we define $\zeta(s, f)=\prod_{i=1}^r \zeta(s, K_i)$.

Theorem A3. *Let $f(X)$ and $g(X)$ be separable primitive polynomials in $Z[X]$. Suppose that $Z(s, f)=Z(s, g)$, which is equivalent to $N(p, f)=N(p, g)$ for all p. Then we have $C(f)=C(g)$, $r(f)=r(g)$ and $\zeta(s, f)=\zeta(s, g)$.*

Proof. Let $K(f)$ and $K(g)$ denote the splitting fields of f and g over Q respectively, and put $K=K(f)K(g)$ and $G=\mathrm{Gal}(K/Q)$. Then $N(p, f)=\mathrm{tr}(\sigma_f(\alpha(p)))$ and $N(p, g)=\mathrm{tr}(\sigma_g(\alpha(p)))$ for $p\nmid D(f)D(g)$ with

$$\sigma_f : G\longrightarrow \mathrm{GL}(\deg(f), C), \qquad \sigma_g : G\longrightarrow \mathrm{GL}(\deg(g), C)$$

and the Frobenius conjugacy class $\alpha(p)\in \mathrm{Conj}(G)$. Hence the Chebotarev density theorem implies $\sigma_f\cong\sigma_g$, so $\zeta(s, f)=L(s, \sigma_f)=L(s, \sigma_g)=\zeta(s, g)$ where $L(s, \sigma_f)$ and $L(s, \sigma_g)$ denote Artin L-functions containing ramified factors.
Q.E.D.

Remark A3. Let $f(X)=f_1(X)\cdots f_r(X)$ be as in the proof of Theorem A1. Put $\pi(t, f)=\#\{1\leq n\leq t ; f_i(n)$ are prime elements in Z for all $i\}$. Then the Hardy-Littlewood conjecture states that

$$\pi(t, f)\sim\frac{C(f)}{\prod_i \deg(f_i)}\cdot\frac{t}{(\log t)^r}\qquad\text{as } t\to\infty$$

when $C(f)\neq 0$ (otherwise $\pi(t, f)$ is constant for large t).

Remark A4. Let $f(X)=X(X+2)(X^4+4)$ and $g(X)=(X^2+4)(X^2+2)(X^2-2)$. Then $Z(s, f)=Z(s, g)$ and $\zeta(s, f)=\zeta(s)^2\zeta(s, Q(\sqrt{-1}, \sqrt{2}))=\zeta(s, Q(\sqrt{-1}))\zeta(s, Q(\sqrt{-2}))\zeta(s, Q(\sqrt{2}))=\zeta(s, g)$. In this case $\prod_i \deg(f_i)=4\neq 8=\prod_i \deg(g_i)$.

Remark A5. For a commutative ring A finitely generated over Z and $f(X_1,\cdots,X_n)\in A[X_1,\cdots,X_n]$ define $Z(s, f)=\prod_p (1-N(p, f)N(p)^{-s})$ where p runs over all maximal ideals of A, $N(p)=\#(A/p)$, and

$$N(p, f)=\#\{(x_1,\cdots,x_n)\in (A/p)^n ; \bar{f}(x_1,\cdots,x_n)=0\}$$

with $\bar{f}$ denoting the reduction modulo p. Then considerations similar to the above ones are possible partially.

Above considerations are concentrated on a truncated spatial zeta function $Z(s, f)$ of the one dimensional arithmetic space $M(f)$, but special values of zeta functions of "non-arithmetical" spaces are also interesting. As an example we note the following simple fact. Let M be a compact Riemann surface of negative Euler-Poincaré characteristic $\chi(M)$. Let $\zeta(s, M)=\prod_p (1-N(p)^{-s})^{-1}$ be the zeta function of M as in [4] and [5] where p runs over all prime closed geodesics on M and $N(p)=\exp(\mathrm{length}(p))$.

Theorem S. *$\zeta(s, M)$ is meromorphic on C with the functional equation*

$$\zeta(s, M)\zeta(-s, M)=(2\sin\pi s)^{2\chi(M)}.$$

Proof. Let

$$Z_0(s, M)=\prod_p \prod_{k=0}^\infty (1-N(p)^{-s-k})$$

denote the original Selberg zeta function of M constructed by Selberg [8], which is the "0-dimensional part" of $\zeta(s, M)$. Then

$$Z_0(s, M)=Z_1(-s, M)\exp\left(-2\chi(M)\int_0^{s-1/2}\pi v\tan\pi v\,dv\right)$$

with $Z_1(s, M) = Z_0(s+1, M)$.   Hence using $\zeta(s, M) = Z_1(s, M)/Z_0(s, M)$ (this being an example of EP decomposition below), we see

$$\zeta(s, M)\zeta(-s, M) = (Z_1(-s, M)/Z_0(s, M))(Z_1(s, M)/Z_0(-s, M))$$

$$= \exp\left(2\chi(M)\left(\int_0^{s-1/2} + \int_0^{-s-1/2}\right)\right)$$

$$= (2\sin \pi s)^{2\chi(M)}. \qquad\qquad\qquad \text{Q.E.D.}$$

This "too simple" functional equation is naturally seen from the view point of multiple (spatial) zeta functions indicated in [5-I] suggesting that Euler products (EP) would have Euler-Poincaré (EP) decompositions. Functional equations of this type hold also for some higher dimensional cases replacing $\zeta(s, M)\zeta(-s, M)$ by $\zeta(s, M)\zeta(a-s, M)^{(-1)^{\dim(M)}}$ in general.   In the case of the m-ple Riemann zeta function $\zeta_m(s)$, the functional equation concerns $\zeta_m(s)\zeta_m(m-s)^{(-1)^m}$.

   Remark S.   Basically $\zeta(s, M)$ describes the closed *strings* (or "morphisms"); we refer to Schwarz [7] for the role of (closed) strings in the SST description of the Universe U.   It seems that the nature of $\zeta(s, U)$ is quite interesting containing the behaviour under the "compactification", zeros, poles, special values... .

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