5. On the Automorphism Groups of a Compact Bordered Riemann Surface of Genus Five

By Kenji NAKAGAWA
NTT, Research Laboratories

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§1. **Introduction.** Let R be a compact bordered Riemann surface of genus g with k boundary components. If $2g+k-1\geqq 2$, the automorphism group of R is a finite group. Then we put $N(g,k)$ to be the maximum order of automorphism groups of R where the maximum is taken over all R of genus g with k boundary components. It is well known that $N(g,k)$ is equal to the maximum order of automorphism groups of Riemann surfaces of genus g deleted k points, and that every automorphism group of R is isomorphic to that of a compact Riemann surface (Oikawa [6]). For every $k\geqq 0$, $N(0,k)$, $N(1,k)$, $N(2,k)$, $N(3,k)$ and $N(4,k)$ are determined by Heins [2], Oikawa [6], Tsuji [7], Tsuji [8] and Kato [4], respectively. In the present paper, we shall determine $N(5,k)$.

§2. **Notation.** Let S be a compact Riemann surface of genus $g\geqq 2$, G be a conformal automorphism group of S and N be the order of G. Let $S_0=S/G$ be the quotient surface with conformal structure induced from S through π, where π is the projection mapping from S onto S_0. Let g_0 be the genus of S_0. At $p\in S$ and at $p_0=\pi(p)\in S_0$, by a suitable choice of local parameters, π is represented locally by $z_0=z^\nu$, where ν is a positive integer, z and z_0 are the local parameters at p, p_0, respectively. If $\nu>1$, p is called a branch point of multiplicity ν. If $\pi(p_1)=\pi(p_2)$ ($p_1, p_2\in S$), then multiplicity at p_1 is equal to that at p_2. Therefore we can define the multiplicity over $p_0\in S_0$ by the multiplicity at $p\in\pi^{-1}(p_0)$. Let t be the number of the points in S_0 which are the projections of all branch points. We call the set of integers g_0 and all multiplicities $\nu_1,\cdots,\nu_t$ the signature of G and denote it by $(g_0;\nu_1,\cdots,\nu_t)$. Without loss of generality, we may assume $\nu_1\leqq\nu_2\leqq\cdots\leqq\nu_t$. For simplicity, we shall denote $(0;\nu_1,\cdots,\nu_t)$ by $(\nu_1,\cdots,\nu_t)$.

§3. **Lemmas.**

Lemma 1 (Wiman [9], Nakagawa [5]). *If ν is a multiplicity of G then $2\leqq\nu\leqq 4g+2$.*

Lemma 2. *There exists neither an automorphism of order 7 nor of order 9 on any compact Riemann surface of genus 5.*

Lemma 3. *For all $k\geqq 0$, $N(5,k)\geqq 8$.*

We are going to determine whether the automorphism group with a given signature exists or not on a compact Riemann surface of genus 5. By Lemma 3, it is not necessary to consider the groups of order $N\leqq 8$. We assume $N>8$. By the Riemann-Hurwitz relation, an easy calculation shows that $g_0 \leqq 1$, $t \leqq 5$. So by Lemma 1, it is enough to consider a finite number of signatures.

§ 4. The existence of hyperelliptic surfaces.

Lemma 4. *Let $\alpha_1, \cdots, \alpha_{2g+2}$ be distinct complex numbers and f be a linear transformation of the sphere which leaves the set $\{\alpha_1, \cdots, \alpha_{2g+2}\}$ invariant. Then there are two automorphisms h_1, h_2 on the hyperelliptic Riemann surface defined by*

$$y^2 = \prod_{n=1}^{2g+2} (x - \alpha_n)$$

such that $f \circ x = x \circ h_j$ $(j=1, 2)$.

Using this lemma, we can show the existence of the following signatures. We shall list up the order N of G, the signature and G_0 (the group of linear transformations of the sphere that leaves $\{\alpha_n\}$ invariant).

N	signature	G_0	N	signature	G_0
120	(2, 3, 10)	icosahedral group I	48	(2, 4, 12)	dihedral group D_{12}
40	(2, 4, 20)	dihedral group D_{10}	24	(2, 12, 12)	cyclic group Z_{12}
24	(4, 4, 6)	dihedral group D_6	24	(2, 2, 3, 3)	tetrahedral group T
22	(2, 11, 22)	cyclic group Z_{11}	20	(2, 20, 20)	cyclic group Z_{10}
20	(4, 4, 10)	dihedral group D_5	12	(2, 3, 4, 4)	dihedral group D_3.

The existence of the groups with signatures (3, 3, 5), (6, 12, 12) is shown in another way.

§ 5. The existence of non-hyperelliptic surfaces. According to Wiman [9], there exist the automorphism groups of order 192, 160, 96 and 64. The signature of the group of order 192 is (2, 3, 8). Then there are a Fuchsian triangle group Γ with signature (2, 3, 8) and the normal subgroup K of Γ of index 192 without elliptic elements such that G is isomorphic to Γ/K. Then $\Gamma = \langle a, b, c \mid a^8 = b^2 = c^3 = abc = id \rangle$, and if we denote by $\bar{a}$, $\bar{b}$ and $\bar{c}$ the K cosets of a, b and c, respectively, then $G = \langle \bar{a}, \bar{b} \rangle$. Thus $\langle \bar{a}, \bar{b}\bar{a}^2\bar{b} \rangle$, $\langle \bar{a}, \bar{b}\bar{a}^4\bar{b} \rangle$ and $\langle \bar{a}, \bar{b}\bar{a}\bar{b}\bar{a}^4\bar{b}\bar{a}\bar{b} \rangle$ are the automorphism group of orders 64, 32 and 16 with signatures (2, 4, 8), (2, 8, 8) and (4, 8, 8), respectively. In the same way, we can show the existence of the groups of orders 96, 96 and 80 with signatures (2, 4, 6), (3, 3, 4) and (2, 5, 5). Moreover, the groups of orders 30 and 15 with signatures (2, 6, 15) and (3, 15, 15) exist.

§ 6. The non-existence of signatures. Now there are Fuchsian groups Γ and K such that G is isomorphic to Γ/K. Then F_K, the Dirichlet region of K, is a finite union of F_Γ. The number of F_Γ's in one F_K is equal to N. Since F_K is symmetric with respect to the rotation $w \to \exp(2\pi i/\nu)w$, there are $N/\nu F_\Gamma$'s in the region $0 \leqq \arg w < 2\pi/\nu$. For example, (3, 3, 11) does not exist. If such a signature existed, the order of corresponding automorphism group would be 33. Three $(=33/11)$ fundamental regions of the Fuchsian group with signature (3, 3, 11) do not form one eleventh part of the fundamental region of any Fuchsian group, since the angle at a vertex of a fundamental region must be $2\pi/m$, where m is an integer. In the same way, we find that (2, 5, 10), (3, 3, 11), (3, 3, 15), (3, 5, 5) and (5, 5, 5) do not exist. Moreover, the non-existence of (2, 3, 12), (2, 3, 22), (2, 5, 6), (3, 4, 12), (5, 5, 15) and (2, 2, 4, 12) is shown.

By summing up above, we obtain

Theorem. *$N(5, k)$ is*

(1) 192 *for* $k\equiv 0, 24, 64, 88 \pmod{96}$

(2) 160 *for* $k\equiv 0, 32 \pmod{40}$ *except the case* (1)

(3) 120 *for* $k\equiv 0, 12, 40, 52 \pmod{60}$ *except the cases* (1), (2)

(4) 96 *for* $k\equiv 16, 32, 40, 48, 56, 72 \pmod{96}$ *except the cases* (2), (3)

(5) 80 *for* $k\equiv 16 \pmod{40}$ *except the cases* (1), (3), (4)

(6) 64 *for* $k\equiv 0 \pmod{8}$ *except the cases* (1)～(5)

(7) 60 *for* $k\equiv 20, 32 \pmod{60}$ *except the cases* (1), (2), (4)～(6)

(8) 48 *for* $k\equiv 0, 4 \pmod{12}$ *except the cases* (1)～(7)

(9) 40 *for* $k\equiv 0, 2 \pmod{10}$ *except the cases* (1)～(8)

(10) 32 *for* $k\equiv 4 \pmod{16}$ *except the cases* (1)～(5), (7)～(9)

(11) 30 *for* $k\equiv 0, 2, 5, 7 \pmod{15}$ *except the cases* (1)～(10)

(12) 24 *for* $k\equiv 2, 6, 10, 14, 20 \pmod{24}$ *except the cases* (1)～(5), (7), (9)～(11)

(13) 22 *for* $k\equiv 0, 1, 2, 3 \pmod{11}$ *except the cases* (1)～(12)

(14) 20 *for* $k\equiv 1, 5, 7, 11 \pmod{20}$ *except the cases* (1)～(8), (10)～(13)

(15) 16 *for* $k\equiv 2, 6 \pmod{16}$ *except the cases* (1)～(5), (7)～(9), (11)～(14)

(16) 15 *for* $k\equiv 1, 6 \pmod{15}$ *except the cases* (1)～(10), (12)～(15)

(17) 12 *for* $k\equiv 0, 1, 3, 4 \pmod{6}$ *except the cases* (1)～(5), (7), (9)～(12)

(18) 8 *otherwise.*

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