43. A Note on the Mean Value of the Zeta and L-functions. III

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1. In the present note we study the twelfth power moment of $L(1/2+it,\chi)$, χ being primitive character mod q. We restrict ourselves to the case of prime q; this is mostly for the sake of simplicity (cf. Remark below).

We consider

$$I=\int_{T-G}^{T+G}\left|L\left(\frac{1}{2}+it,\chi\right)\right|^2 dt,$$

where

$$(1)\qquad q^{1/2}\leqq T,\quad (qT)^{\varepsilon}\leqq G\leqq (qT)^{1/3}l^{-1}\quad (l=\log qT).$$

Using the function E_1 introduced in [4], we have

$$I\ll Gl+\left|\int_{-\infty}^{\infty}E_1(T+t,\chi)tG^{-2}\exp(-(t/G)^2)dt\right|.$$

Then following closely the argument of [4, §2] one may show that for an $N\approx qT$

$$\begin{aligned}I\ll\ & Gl+G^{-1}((qT)^{1/4}+q^{1/2}(qT)^{\varepsilon})l\\ &+G\left|\sum_{n\leqq N}a(n,\chi)\int_0^{\infty}(y(y+1))^{-1/2}\cos(T\log(1+1/y))\right.\\ &\left.\times\exp\left(-2\pi iny/q-\frac{1}{4}(G\log(1+1/y))^2\right)dy\right|.\end{aligned}$$

To estimate this sum over n, we divide it into two parts according to $qTG^{-2}l^{-2}<n\leqq N$ and $n\leqq qTG^{-2}l^{-2}$. To the integrals in the first sum we apply the second mean value theorem, and find that they are $\ll ql(nG)^{-1}$. Thus by [4, Lemma 5] we see that the first sum is $\ll q^{1/2}G^{-1}l^3$. On the other hand, to the integrals in the second sum we apply the saddle point method, and after overcoming somewhat lengthy computation we find that they are equal to

$$\begin{aligned}&\pi^{1/4}q^{1/2}(\pi n^2+2qTn)^{-1/4}\\ &\times\exp\left(-2iTF\left(\frac{\pi n}{2qT}\right)+\frac{\pi in}{q}-\frac{\pi i}{4}-\left(G\sinh^{-1}\left(\frac{\pi n}{2qT}\right)\right)^2\right)+O((q/nT)^{1/2}),\end{aligned}$$

where

$$F(x)=\sinh^{-1}(x^{1/2})+(x(x+1))^{1/2}.$$

These error terms contribute to the sum the amount of $\ll q^{1/2}G^{-1}l^2$, because of [4, Lemma 5].

Collecting these and using partial summation, we get

Lemma 1. *On the condition* (1) *we have*

$$I \ll Gl + q^{1/2}l^3 + (qT)^{1/4}G^{-1}l$$
$$+ G(q/T)^{1/4} \sum_K K^{-1/4}\left(|S(K, K, T)| + K^{-1}\int_0^K |S(x, K, T)|dx\right)\exp(-G^2K/qT),$$

where $(qT)^{1/3} \leqq K = 2^j \leqq qTG^{-2}l^{-2}$ *and*

$$S(x, K, \tau) = \sum_{K \leqq n < K+x} a(n, \chi)\exp\left(-2i TF\left(\frac{\pi n}{2q\tau}\right) + \frac{\pi i n}{q}\right).$$

2. Hereafter we follow the argument of Heath-Brown [1] (cf. Ivić [3, § 8.3]). First we need a mean-value theorem for $S(x, K, \tau)$ over well-spaced points τ:

Lemma 2. *Let* $\mathcal{S}$ *be a set of real numbers* τ_r *such that* $T/2 \leqq \tau_r \leqq T$ *and* $(qT)^{\varepsilon} \leqq G \leqq |\tau_r - \tau_s| \leqq J$ *for* $r \neq s$. *Then for* $K \leqq qTl^{-1}$ *and* $0 < x \leqq K$, *we have*

$$\sum_{\tau_r \in \mathcal{S}} |S(x, K, \tau_r)| \ll (Kq^{-1/2}l^{3/2} + K^{3/4}T^{1/4}q^{-1/4}G^{-1/2}l)|\mathcal{S}|^{1/2} + J^{1/4}K^{5/8}T^{-1/8}q^{-3/8}l^{3/2}|\mathcal{S}|.$$

The proof is the same as that of [3, Lemma 8.1] except for the fact that here we have to appeal to [4, Lemma 5].

We may now study the distribution of the large values of $L(1/2+it, \chi)$. Thus let $\mathcal{A} = \{t_r\}$ be the set of real numbers such that $T/2 \leqq t_r \leqq T$, $|t_r - t_s| \geqq 1$ for $r \neq s$, and $|L(1/2+it, \chi)| \geqq V$. We assume that $T \geqq q^{1/2}$ and

$$(2) \qquad V \geqq \mathrm{Max}\,(q^{1/4}, (qT)^{1/16})l^3.$$

We set $G = V^2 l^{-6}$. Then (1) is satisfied, for by [4, Theorem] we have $V \leqq (qT)^{1/6}l^{5/2}$ always. Further we set $J = G^3 q^{-1}$ so that $G \leqq J \leqq T$, since we have (1) and (2). Next, let $\mathcal{A}_I$ be the subset of $\mathcal{A}$ contained in an interval I of length J. We divide I into sub-intervals of length G, and let $\mathcal{B}$ be the set of the middle points of those sub-intervals containing the points of $\mathcal{A}$. Then we have, by the argument of [2, § 4],

$$|\mathcal{A}_I| V^2 \ll l \sum_{\tau \in \mathcal{B}} \int_{\tau - G}^{\tau + G} \left| L\left(\frac{1}{2} + it, \chi\right)\right|^2 dt.$$

Hence, on noting (2), we have, by Lemma 1,

$$|\mathcal{A}_I| \ll (q/T)^{1/4} l^{-5} \sum_K K^{-1/4} \sum_{\tau \in \mathcal{B}} \left(|S(K, K, \tau)| + K^{-1}\int_0^K |S(x, K, \tau)|dx\right)$$
$$\times \exp(-G^2K/qT).$$

And by Lemma 2 we find that

$$|\mathcal{A}_I| \ll qTG^{-3}l^{-7} + (JG^{-3}ql^{-14})^{1/4}|\mathcal{B}|.$$

This and the present choice of J imply that $|\mathcal{A}_I| \ll qTG^{-3}l^{-7}$. Adding over T/J intervals I we get

$$|\mathcal{A}| \ll (qT)^2 V^{-12} l^{29}.$$

This settles the case when (2) holds. If (2) does not hold, then we appeal to [4, Theorem], which implies

$$|\mathcal{A}| \ll Tl^5 V^{-2} \ll (qT)^2 V^{-12} l^{35}.$$

Therefore we have obtained

Theorem. *Let* $\mathcal{A}$ *be the set defined as above. Then we have*

$$|\mathcal{A}| \ll (qT)^2 V^{-12} l^{35}.$$

Corollary. *Let* χ *be a non-principal character* mod q, *a prime. Then* *we have, for* $T \geqq 1$,

$$\int_0^T \left| L\left(\frac{1}{2}+it, \chi\right) \right|^{12} dt \ll ((qT)^2+q^3)(\log qT)^{36}.$$

Remark. We may treat the case of composite moduli as well, but with a further complexity caused by the coefficients $a(n, \chi)$. For this sake [4, Lemma 5] should be replaced by

$$\sum_{n \leqq N} |a(n, \chi)| \ll (q^{-1/2}N + q^{-1/4}N^{1/8})(qN)^{\varepsilon},$$
$$\sum_{n \leqq N} |a(n, \chi)|^2 \ll (q^{-1}N + q^{-1/2}N^{1/4})(qN)^{\varepsilon},$$

where $q>1$ is arbitrary and $\chi \pmod q$ is primitive. These can be proved by a modification of the argument of Heath-Brown [2].

References

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