32. *Relative Zariski Decomposition on Higher Dimensional Algebraic Varieties*

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0. Introduction. The purpose of this note is to state several results in my Master Thesis [7]. The details will be published elsewhere. The main theorem of this note is Theorem 3. By this theorem, if K_X has a good Zariski decomposition, then the canonical ring of X is finitely generated. Theorem 1 and Theorem 2 are key theorems to prove Theorem 3. Theorem 5 is a characterization of a nef and good divisor by μ_x. All varieties in this note are assumed to be defined over an algebraically closed field of characteristic zero.

1. Notation. Let X be an algebraic scheme. We denote the group of Cartier divisors on X by $\mathrm{Div}(X)$. For a non-zero rational function ϕ on X, the principal Cartier divisor defined by ϕ is denoted by $\mathrm{div}(\phi)$. For $D_1, D_2 \in \mathrm{Div}(X)\otimes \boldsymbol{R}$, we say D_1 is *$\boldsymbol{R}$-linear equivalent* to D_2, which is denoted by $D_1 \sim_{\boldsymbol{R}} D_2$, if there exists a positive integer m and exists a non-zero rational function ϕ on X such that $D_1 = D_2 + (1/m)\,\mathrm{div}(\phi)$. For a real number a, the lounding-up, the lounding-down, the nearest integer and the fractional part of a are denoted by $\ulcorner a \urcorner$, $[a]$, $\langle a \rangle$ and $\{a\}$ respectively, where in case $\{a\}=1/2$, we define $\langle a \rangle = \ulcorner a \urcorner$ if $a>0$, $\langle a \rangle = [a]$ if $a<0$. From now on, we assume X is non-singular. Let D be an element of $\mathrm{Div}(X)\otimes \boldsymbol{R}$ and $D=\sum_i a_i D_i$ the irreducible decomposition of D. Then we set $\ulcorner D \urcorner = \sum_i \ulcorner a_i \urcorner D_i$, $[D]=\sum_i [a_i]D_i$, $\langle D \rangle = \sum_i \langle a_i \rangle D_i$ and $\{D\} = \sum_i \{a_i\} D_i$. Let $\mathcal{J}$ be an ideal sheaf of $\mathcal{O}_X$ and x a point of X (not necessarily closed). Then we define

$$\mathrm{ord}_x(\mathcal{J}) = \max\{a \in \boldsymbol{N} \cup \{\infty\} \mid \mathcal{J}\mathcal{O}_{X,x} \subseteqq m_x^a\} \text{ and } \mathrm{ord}_x(D) = \sum_i a_i\, \mathrm{ord}_x(\mathcal{O}_X(-D_i)),$$

where m_x is the maximal ideal of $\mathcal{O}_{X,x}$. We furthermore assume X is complete. We set $\kappa(X, D) = \max_m \{\kappa(X, [mD])\}$. If $\kappa(X, D) = \dim X$, D is called *big*. D is called *good* if there exists a birational morphism $\pi : Y \to X$ of non-singular complete varieties and exists a fiber space $h : Y \to Z$ of non-singular complete varieties such that $\pi^*(D) \sim_{\boldsymbol{R}} h^*(M)$ for some big element M of $\mathrm{Div}(Z)\otimes \boldsymbol{R}$. Next, we consider the relative case. Let X be a non-singular algebraic variety, S an algebraic variety, $f: X \to S$ a proper surjective morphism. For $D \in \mathrm{Div}(X)\otimes \boldsymbol{R}$, we set

$$E(X/S, D) = \{n \in \boldsymbol{N}\backslash\{0\} \mid f_*\mathcal{O}_X([nD]) \neq 0\}.$$

D is called *f-nef* if $(D\cdot C) \geqq 0$ for any complete curve C on X such that $f(C)$ is a point. D is called *f-big* (resp. *f-good*) if $D|_{X_\eta}$ is a big (resp. good) element of $\mathrm{Div}(X_\eta)\otimes \boldsymbol{R}$, where X_η is the generic fiber of f. For a Cartier divisor H on X, the *f-base locus of H* $\mathrm{Bs}(X/S, H)$ is defined by

$$\mathrm{Bs}(X/S, H)=\mathrm{Supp}(\mathrm{Coker}(f^*f_*\mathcal{O}_X(H)\longrightarrow\mathcal{O}_X(H))).$$

If $\mathrm{Bs}(X/S, H)=\phi$, H is called *f-free*. An element L of $\mathrm{Div}(X)\otimes\boldsymbol{Q}$ is called *f-semi-ample* if there exists a positive integer m such that $mL\in\mathrm{Div}(X)$ and mL is f-free. For $D\in\mathrm{Div}(X)\otimes\boldsymbol{R}$, a decomposition $D=P+N$ is called an *f-sectional decomposition* if $P, N\in\mathrm{Div}(X)\otimes\boldsymbol{R}$, N is effective and there exists a positive integer d such that the natural homomorphism $f_*\mathcal{O}_X([ndP])\to f_*\mathcal{O}_X([ndD])$ is bijective for every $n\geqq 0$. P (resp. N) is called *the positive part* (resp. *negative part*) of this decomposition. An f-sectional decomposition $D=P+N$ is called an *f-Zariski decomposition* (resp. *good f-Zariski decomposition*) if the positive part P is f-nef (resp. f-nef and f-good). Let D be an element of $\mathrm{Div}(X)\otimes\boldsymbol{R}$ and x a point of X (not necessarily closed). We set

$$\mathcal{I}_n(X/S, D)=\mathrm{Im}(f^*f_*\mathcal{O}_X([nD])\otimes\mathcal{O}_X(-[nD])\longrightarrow\mathcal{O}_X)$$

and

$$\mu_x(X/S, D)=\inf_n((\mathrm{ord}_x(\mathcal{I}_n(X/S, D))+\mathrm{ord}_x(\{nD\}))/n).$$

By the definition of $\mu_x(X/S, D)$, $\mu_x(X/S, D)$ is upper semi-continuous with respect to $x\in X$.

2. Non-vanishing theorem and vanishing theorem. We refer the reader to [3] for the notion concerning generalized normal crossing varieties.

Theorem 1 (Non-vanishing theorem). *Let X be a generalized normal crossing variety, Z a projective variety and let $f: X\to Z$ be a morphism. Let D_j be an element of* $\mathrm{Div}(Z)$ *and d_j a real number for every $j\in J$, where J is a finite subset of $\boldsymbol{N}$. We assume the following.*

(i) *For all $n\geqq 0$, every connected component of X_n is mapped surjectively to Z.*

(ii) *$D=\sum_{j\in J}d_jD_j$ is nef.*

(iii) *There exists an element A of* $\mathrm{Div}_0(X)\otimes\boldsymbol{R}$ *such that the support of A is a generalized normal crossing divisor on X and $\ulcorner A\urcorner\geqq 0$.*

(iv) *There exists a positive number t_0 and exists an ample element L of* $\mathrm{Div}(Z)\otimes\boldsymbol{R}$ *such that $t_0f^*(D)+A-K_X\sim_{\boldsymbol{R}}f^*(L)$.*

Then there are positive numbers t_1 and ε_1 such that for any $t\geqq t_1$ satisfying $|\langle td_j\rangle-td_j|<\varepsilon_1$ for all $j\in J$, we have

$$H^0(X, \mathcal{O}_X(f^*(\textstyle\sum_{j\in J}\langle td_j\rangle D_j)+\ulcorner A\urcorner))\neq 0.$$

Theorem 2 (Vanishing theorem). *Let X be a non-singular algebraic variety, S an algebraic variety and let $f: X\to S$ be a proper surjective morphism. Let L be an element of* $\mathrm{Div}(X)\otimes\boldsymbol{R}$ *such that L is f-nef and f-good and $\{L\}_{\mathrm{red}}$ has only normal crossings. Let E, E' be elements of* $\mathrm{Div}(X)$ *such that E and E' are effective and $E+E'\in|[mL]|$ for some positive integer m. Then homomorphisms induced by the natural homomorphism $\mathcal{O}_X\to\mathcal{O}_X(E)$*

$$\phi_E^i: R^if_*\mathcal{O}_X(K_X+\ulcorner L\urcorner)\longrightarrow R^if_*\mathcal{O}_X(K_X+\ulcorner L\urcorner+E)$$

are injective for all $i\geqq 0$.

Theorem 1 is a generalization of [3, Theorem 5.1] and [4, Theorem 3]. Theorem 2 is a relative version of [5].

3. Rationality and semi-ampleness.

Theorem 3. *Let X be a non-singular algebraic variety, S an algebraic variety and let $f: X \to S$ be a proper surjective morphism. Let Δ be an element of* $\mathrm{Div}(X) \otimes \boldsymbol{Q}$ *such that* $[\Delta]=0$ *and* Δ_{red} *has only normal crossings. We assume that $K_X+\Delta$ has a good f-Zariski decomposition $K_X+\Delta=P+N$, where P is the positive part of this decomposition. Then* $P \in \mathrm{Div}(X) \otimes \boldsymbol{Q}$ *and P is f-semi-ample.*

Theorem 3 is a generalization of [4, Theorem 1]. Using Theorem 3, we have that $R(X, K_X+\Delta)=\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X([n(K_X+\Delta)]))$ is finitely generated if X is complete and $\kappa(X, K_X+\Delta) \leqq 2$. (cf. [6, Theorem (3, 1)].) We remark that Cutkosky [1] gave an example of a big divisor which has no Zariski decomposition with rational coefficients.

4. *f*-sectional decomposition. Let X be a non-singular variety, S an algebraic variety and $f: X \to S$ be a proper surjective morphism. Let D be an element of $\mathrm{Div}(X) \otimes \boldsymbol{R}$ such that $E(X/S, D) \neq \phi$. Then it is easy to see that there are a finite number of prime divisors Γ such that $\mu_\Gamma(X/S, D) > 0$. Hence we can set

$$N(X/S, D)=\sum_{\Gamma:\text{ prime divisors}} \mu_\Gamma(X/S, D)\Gamma \quad \text{and} \quad P(X/S, D)=D-N(X/S, D).$$

Proposition 4. *Notation being the same as above, we have*

(i) *$D=P(X/S, D)+N(X/S, D)$ is an f-sectional decomposition,*

(ii) *for any f-sectional decomposition $D=P+N$,*

$$\mu_x(X/S, D)=\mu_x(X/S, P)+\mathrm{ord}_x(N)$$

for all $x \in X$, and

(iii) *for any f-sectional decomposition $D=P+N$, $N \leqq N(X/S, D)$.*

We call the f-sectional decomposition $D=P(X/S, D)+N(X/S, D)$ *the canonical f-sectional decomposition.*

Theorem 5. *Let X, S and f be the same as in Proposition 4. For* $L \in \mathrm{Div}(X) \otimes \boldsymbol{R}$, *the following are equivalent.*

(i) *$\mu_x(X/S, L)=0$ for all $x \in X$.*

(ii) *L is f-nef and f-good.*

Theorem 5 means that L is almost base point free in the sense of Goodman [2] if and only if L is nef and good.

Corollary 6 (Uniqueness of the good Zariski decomposition). *Let X, S and f be the same as in Proposition 4. Let D be an element of* $\mathrm{Div}(X) \otimes \boldsymbol{R}$ *and $D=P+N$ a good f-Zariski decomposition. Then $P=P(X/S, D)$ and $N=N(X/S, D)$.*

References

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