30. A Topology on Arithmetical Lattice-Ordered Groups[†]

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(Communicated by Shokichi IYANAGA, M. J. A., March 12, 1986)

The aim of this note is to describe explicitly the weakest topology on an arithmetical lattice-ordered group for which given lattice-ideal is open. This topology is utilized to treat some of ring topologies, field topologies and $\mathfrak{p}$-adic topologies.

1. A lattice-ordered group (abbr. *l.o.* group) $G=(G, \cdot, \leq)$ is called *arithmetical*, if it is a conditionally complete lattice and a free group generated by the set P of all prime elements in the cone (integral part) of G. Then G is abelian by Iwasawa's theorem for *l.o.* groups [1], [2], so that each element a of G has a unique factorization in the form:

$$a=\prod_{p\in P} p^{\nu(p,a)}, \qquad \nu(p,a)\in \boldsymbol{Z}$$

where $\boldsymbol{Z}$ is the integers and $\nu(p, a)$ is the exponent of a at p. This factorization was generalized by the author [4] as follows. Each lattice-ideal (abbr. *l*-ideal) J of G has a unique factorization in the form:

$$J=\prod_{p\in P_+(J)} J(p)^{\nu(p,J)}\cdot \bigcup \prod_{p\in P_-(J)} J(p)^{\nu(p,J)}\cdot e_{P_{-\infty}(J)}$$

where $\prod$ is finite product, $\cup$ is set-theoretical union, $J(p)$ is the principal *l*-ideal generated by p, $\nu(p, J)=\inf\{\nu(p, a); a\in J\}$, $P_+(J)=\{p\in P; 0<\nu(p, J)\}$, $P_-(J)=\{p\in P; -\infty<\nu(p, J)<0\}$, $P_{-\infty}(J)=\{p\in P; \nu(p, J)=-\infty\}$, $e_{P_{-\infty}(J)}$ is the $P_{-\infty}(J)$-component of unit e of G. (If J is principal, $P_-(J)$ is finite and $e_{P_{-\infty}(J)}$ coincides with the cone of G.)

A non-void set $\boldsymbol{U}$ of *l*-ideals of G is called a *u-system* of G if it satisfies the following conditions:

1) If $J_1, J_2\in \boldsymbol{U}$, there is $J_3\in \boldsymbol{U}$ such that $J_3\subseteq J_1\cap J_2$.

2) If $a\in G$, $J_1\in \boldsymbol{U}$, there is $J_2\in \boldsymbol{U}$ such that $aJ_2\subseteq J_1$.

3) If $J_1\in \boldsymbol{U}$, there is $J_2\in \boldsymbol{U}$ such that $J_2J_2\subseteq J_1$.

Then $\boldsymbol{U}$ determines a topology on G, which is called an *l-ideal topology* on G. In symbol: $\mathrm{T}(\boldsymbol{U})$. Let $g(n; p, J)$ be the integer m or $-\infty$ such that $\nu(p, J)/2^n\leq m<\nu(p, J)/2^n+1$. We define

$$J^{(n)}=\bigcup \prod_{p\in P_-(J)} J(p)^{g(n;p,J)}\cdot e_{P_{-\infty}(J)}$$

for $n\in \boldsymbol{N}_0$, the non-negative integers. Then since (1°) $J^{(n)}\supseteq J^{(n+1)}$, (2°) $J^{(n)}\supseteq e_{P_{-\infty}(J)}$, (3°) $J^{(n)}\supseteq J^{(n+1)}J^{(n+1)}$ and (4°) $(J^{(n)})^{(m)}=J^{(n+m)}$, we can show that

$$\boldsymbol{U}(J)=\{aJ^{(n)}; a\in G, n\in \boldsymbol{N}_0\}$$

forms a u-system of G.

Theorem 1. *Let J be an l-ideal of G. Then among the set of all*

[†] Dedicated to Emeritus Professor Hidetaka TERASAKA for his octogenarian birthday.

l-ideal topologies for which J is open, there exists the weakest one, and it is given by $\mathrm{T}(\boldsymbol{U}(J))$.

Proof. It is obvious that J is open for $\mathrm{T}(\boldsymbol{U}(J))$. Let T_J be any l-ideal topology for which J is open. We may assume that J is a member of a u-system $\boldsymbol{U}$ which determines T_J. Then we can take an l-ideal K such that $K^{2n}\subseteq J$, $K\in\boldsymbol{U}$. Since $2^n\nu(p, K)=\nu(p, K^{2n})\geq\nu(p, J)$, we have $\nu(p, K)\geq g(n; p, J)$ for all p, so that there is an l-ideal $K=K_n\in\boldsymbol{U}$ with $K_n\subseteq J^{(n)}$ for each $n\in\boldsymbol{N}_0$. Thus for any $aJ^{(n)}$ we have $I\subseteq aK_n\subseteq aJ^{(n)}$, choosing $I\in\boldsymbol{U}$ as $a^{-1}I\subseteq K_n$. This means that $\mathrm{T}(\boldsymbol{U}(J))$ is weaker than T_J. Q.E.D.

An l-ideal J of G is called *bounded* for a u-system $\boldsymbol{U}$, if for each $K\in\boldsymbol{U}$ there is $I\in\boldsymbol{U}$ such that $IJ\subseteq K$. A u-system $\boldsymbol{U}$ is called *locally bounded* if it contains at least one member bounded for $\boldsymbol{U}$. For a u-system $\boldsymbol{U}$ the set of all l-ideals J' with $J'\supseteq J$ for some $J\in\boldsymbol{U}$ will be denoted by $\mathcal{F}(\boldsymbol{U})$. Then we have

Theorem 2. *The following properties are equivalent:*

(1) *$\boldsymbol{U}(J)$ is locally bounded.*

(2) *$e_{P_{-\infty}(J)}=J^{(n)}$ for some $n\in\boldsymbol{N}_0$.*

(3) $\mathcal{F}(\boldsymbol{U}(J))=\mathcal{F}(\boldsymbol{U}(e_{P_{-\infty}(J)}))$.

Proof. By using (4°) we can show (1)$\Rightarrow$(2). (2)$\Rightarrow$(3) follows from the fact that $K\in\boldsymbol{U}(J)\Rightarrow\mathcal{F}(\boldsymbol{U}(K))=\mathcal{F}(\boldsymbol{U}(J))$. Evidently $\boldsymbol{U}(e_Q)$ is locally bounded for any subset Q of $\boldsymbol{P}$. Hence (3)$\Rightarrow$(1) is obvious.

Theorem 3. *If $\mathcal{F}(\boldsymbol{U})=\boldsymbol{U}$, then $\boldsymbol{U}$ is locally bounded if and only if $\boldsymbol{U}=\mathcal{F}(\boldsymbol{U}(e_Q))$ for a subset Q of $\boldsymbol{P}$.*

Proof. "If part" is immediate by Theorem 2. We can see that J is bounded for $\boldsymbol{U}$ if and only if for any $K\in\boldsymbol{U}$ there is $a\in G$ such that $aJ\subseteq K$. Accordingly we have $\mathcal{F}(\boldsymbol{U}(J))\supseteq\mathcal{F}(\boldsymbol{U}(K))$, hence

$$\mathcal{F}(\boldsymbol{U}(J))\supseteq\mathcal{F}(\boldsymbol{U})=\cup\{\mathcal{F}(\boldsymbol{U}(K))\,;\,K\in\boldsymbol{U}\}\supseteq\mathcal{F}(\boldsymbol{U}(J))$$

and hence $\boldsymbol{U}=\mathcal{F}(\boldsymbol{U}(J))=\mathcal{F}(\boldsymbol{U}(e_{P_{-\infty}(J)}))$. This proves the "only if part" of the theorem.

2. Let R be a (not necessarily commutative) ring with unity quantity, $\mathfrak{O}$ a regular Asano order of R, $\boldsymbol{G}$ the *l.o.* group of all fractional two-sided $\mathfrak{O}$-ideals in R, and $\mathfrak{M}$ the *l.o.* semigroup of all two-sided $\mathfrak{O}$-submodules of R, each of which contains at least one regular element of R. For each $M\in\mathfrak{M}$, $f(M)=\{\mathfrak{a}\in\boldsymbol{G}\,;\,\mathfrak{a}\subseteq M\}$ is an l-ideal of $\boldsymbol{G}$, and conversely for any l-ideal $\mathcal{J}$ of $\boldsymbol{G}$, $\mathcal{J}\mapsto\cup\{\mathfrak{a}\in\boldsymbol{G}\,;\,\mathfrak{a}\in\mathcal{J}\}$ is the inverse of f. We see readily that f is an isomorphism from $\mathfrak{M}$ to the multiplicative *l.o.* semigroup of all l-ideals of $\boldsymbol{G}$ as *l.o.* semigroups. Then we see that a subset $\mathfrak{U}$ of $\mathfrak{M}$ forms a fundamental system of neighbourhoods of zero if and only if $\{f(M)\,;\,M\in\mathfrak{U}\}$ is a u-system of $\boldsymbol{G}$, where for $\mathfrak{U}$ we employ the five axioms (II), ($\mathrm{III_a}$), ($\mathrm{III_b}$), ($\mathrm{IV_a}$) and ($\mathrm{IV_b}$) in [3]. The ring topology [3], [5] determined by the above $\mathfrak{U}$ is called here *$\mathfrak{O}$-topology* on R. In symbol: $\mathrm{T}(\mathfrak{U})$. For each $M\in\mathfrak{M}$ we put

$$\mathfrak{U}(M)=\{\mathfrak{a}M^{(n)}\,;\,\mathfrak{a}\in\boldsymbol{G},\ M^{(n)}=\bigcup_{\mathfrak{p}\in P_-(M)}\prod\mathfrak{p}^{g(n;\,\mathfrak{p},\,M)}\mathfrak{O}_{P_{-\infty}(M)},\ n\in\boldsymbol{N}_0\}.$$

Then $\mathfrak{U}(M)$ is a fundamental system of neighbourhoods of zero. By Theorem 1, $\mathrm{T}(\mathfrak{U}(M))$ is an $\mathfrak{O}$-topology on R, and it is the weakest $\mathfrak{O}$-topology among the set of all $\mathfrak{O}$-topologies for which M is open. Moreover we see, by Theorem 2, that the following statements are equivalent:

(1) $\mathrm{T}(\mathfrak{U}(M))$ *is locally bounded.*

(2) $P_{-\infty}(M)$*-component of* $\mathfrak{O}$ *is* $M^{(n)}$ *for some* $n \in N_0$.

(3) $\mathrm{T}(\mathfrak{U}(M)) = \mathrm{T}(\mathfrak{U}(\mathfrak{O}_{P_{-\infty}(M)}))$.

It is then readily seen that *an* $\mathfrak{O}$*-topology* T *on* R *is locally bounded if and only if* $\mathrm{T} = \mathrm{T}(\mathfrak{O}_Q)$ *for a set* Q *of prime* $\mathfrak{O}$*-ideals.*

If in particular R is a quotient field of a Dedekind domain $\mathfrak{O}$, the following statements are equivalent:

(1) $\mathrm{T}(\mathfrak{U}(M))$ *is the supremum of a finite number of* $\mathfrak{p}$*-adic topologies on* R.

(2) $\mathrm{T}(\mathfrak{U}(M))$ *is a field topology* [3], [5] *on* R, *that is for each non-zero* $x \in R$ *and each open* $\mathfrak{O}$*-submodule* M, *there is an open* $\mathfrak{O}$*-submodule* M' *such that* $(x+M')^{-1} \subseteq x^{-1} + M$.

(3) M *is a non-zero fractional* $\mathfrak{O}_{P_{-\infty}(M)}$*-ideal in* R.

(4) $P_{-}(M)$ *is a finite set.*

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