23. The Grothendieck Conjecture and Padé Approximations*)

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§0. The Grothendieck conjecture [1], [2] predicts the global algebraic behavior of solutions of linear differential equations, provided that these equations have "sufficiently many solutions" after reduction (mod p) for almost all p. In-depth studies of this conjecture and its interesting generalizations belong to Katz [1], [3]. However, the conjecture remains open in many important cases. One of the crucial cases, pointed out in [1], [4], is the case of Lamé-type equations or the case of rank one equations over an elliptic curve. In this case, we show how the methods of Padé approximations can be used to prove the Grothendieck conjecture in this and other important cases.

§1. For expositions of the p-adic properties of linear differential equations connected with the Grothendieck conjecture see [1], [2], [3], [4]. If a linear differential equation is represented in a matrix form

$$(1)\qquad (d/dx)\vec{f}+A(x)\vec{f}=0,$$

with $A\overset{\text{def}}{=}A(x)\in M(n,K(x))$ and an algebraic number field K, then the p-curvature operator Ψ_p of (1) mod p is $\Psi_p=((d/dx)\cdot I+A)^p (\text{mod } p)$.

Here Ψ_p is, in fact, a linear operator: $\Psi_p=A_p$ (mod p), where $A_1=A$, $A_{n+1}=(d/dx)A_n+AA_n$.

The Grothendieck conjecture. For a system (1), $\Psi_p=0$ for almost all p if and only if all solutions of (1) are algebraic functions. Detailed studies of equivalents of the Grothendieck conjecture are presented in Honda [2] for scalar linear differential equations

$$(2)\qquad Lf\overset{\text{def}}{=}a_n f^{(n)}+\cdots+a_1 f'+a_0 f=0$$

and $a_i=a_i(x)\in K[x]$ $(0\le i\le n)$. Let, for a prime ideal ρ of K, $\overline{K}_\rho$ denotes the residue field and L_ρ denotes the reduction mod ρ of L. Other reformulations of the Grothendieck conjecture are the following:

1) If for almost all prime ideals ρ, $L_\rho f=0$ has n solutions in $\overline{K}_\rho(x)$ which are independent over $\overline{K}_\rho(x^p)$, then all solutions of (2) are algebraic functions;

2) If for almost all p we have $(d/dx)^p\equiv 0 \text{ mod } K_\rho((x))[d/dx]L_\rho$, then all solutions of (2) are algebraic functions.

A condition weaker than assumptions of the Grothendieck conjecture is the condition of global nilpotence of (1), i.e., the condition of nilpotence of matrices Ψ_p for almost all p [1], [2].

*) To Professor Bers on his 70th Birthday.

§2. We assume below that $f(x)$ satisfies a local version of the assumptions of the Grothendieck conjecture. Namely, for an algebraic number field K and some $\zeta \in K$ we assume that all functions $f^i(x)$ have Taylor expansions $f^i(x)=\sum_{n=0}^{\infty} a_{n,i}(x-\zeta)^n$ with $a_{n,i} \in K$ $(n=0,1,\cdots; i=0, 1,\cdots,m-1)$ and we assume that for some $c_0 \geq 1$ (depending only on $f(x)$ and ζ),

(3) $\quad |\overline{a_{n,i}}| \leq c_0^n : i=0,1,\cdots,$ den $\{a_{0,i,\ldots,}\ a_{n,i} : i=0,1,\cdots\} \leq c_0^n : n \geq n_0(m).$

Here $|\overline{\alpha}|$ is a size of an algebraic number $\alpha \in K$, i.e. $|\overline{\alpha}|=\max\{|\alpha^{(\sigma)}|: \sigma=1,\cdots,d\}$ where $\alpha^{(\sigma)}$ are all numbers algebraically conjugate to α; and den $\{\alpha_0,\cdots,\alpha_n\}$ denotes the common denominator of algebraic numbers $\alpha_0,\cdots,\alpha_n$. We need an auxiliary

Lemma 1. *Let M, N be integers $N>M>0$ and let u_{ij} $(1 \leq i \leq M, 1 \leq j \leq N)$ be algebraic integers in K with sizes at most $U(\geq 1)$. Then there exist algebraic integers $x_1,\cdots,x_N$ in K, not all 0, satisfying $\sum_{j=1}^{N} u_{ij}\cdot x_j=0$ $(1 \leq i \leq M)$ and $|\overline{x}_j| \leq c_1(c_1NU)^{M/(N-M)}$ $(1 \leq j \leq M)$. Here $c_1=c_1(K)>0$.*

The existence of Padé approximations to $1, f(x),\cdots,f^{m-1}(x)$ at $x=\zeta$ is given by

Lemma 2. *Let $1>\varepsilon>0$ and D be a sufficiently large integer, $D \geq D_0(f, m, \zeta, \varepsilon)$.*

Under the assumptions above, there exist polynomials $P_0(x),\cdots, P_{m-1}(x) \in K[x]$ not all zero of degree at most D with integer coefficients of sizes bounded by c_2^{mD}, $c_2=c_2(f,K,\zeta,\varepsilon)>0$ and such that the function

$$R(x)=\sum_{i=0}^{m-1} P_i(x)f(x)^i$$

has a zero at $x=\zeta$ of order at least $mD-[\varepsilon mD]$.

Proof. Let $P_i(x)=\sum_{n=0}^{D} p_{n,i}\cdot(x-\zeta)^n$, where $p_{n,i}$ are undetermined integers from K $(i=0,\cdots,m-1; n=0,\cdots,D)$. Then, in the notation above, $R(x)=\sum_{n=0}^{\infty} x^n\cdot\{\sum_{i=0}^{m-1}\sum_{k=0,k\leq n}^{D} p_{k,i}a_{n-k,i}\}$. Then the system of linear equations on $p_{n,i}$, equivalent to the condition $\mathrm{ord}_{x=\zeta} R(x) \geq M \overset{\mathrm{def}}{=} mD-[\varepsilon mD]$, has the form:

(4) $$\sum_{i=0}^{m-1}\sum_{k=0,k\leq n}^{D} p_{k,i}\cdot a_{n-k,i}=0: \quad n=0,1,\cdots,M-1.$$

This is a system of M equations in $m(D+1)>M$ unknowns $p_{n,i}$ $(i=0,\cdots,m-1; n=0,\cdots,D)$ with coefficients of sizes at most c_0^M and a common denominator bounded by c_0^M (according to (3)). Applying Lemma 1, we obtain a nontrivial solution of (4) in integers $p_{n,i}$ from K of sizes bounded by $c_3^{mD(1+\varepsilon)2/\varepsilon}$, where $c_3=c_3(c_0,K)>0$. Then the corresponding polynomials $P_i(x)$ $(i=0,\cdots,m-1)$ satisfy all the conditions of Lemma 2.

Let us assume for now that $R(x)\not\equiv 0$. According to the expansion of $R(x)$ in proof of Lemma 2 we have $R(x)=c_r(x-\zeta)^r+0((x-\zeta)^{r+1})$, where $c_r\neq 0$ is an algebraic number from K of size at most $c_2^{mD}\cdot c_4^r$, with the denominator bounded by c_4^r, $c_4=c_4(c_0)>0$.

Lemma 3. *Let us assume that there is a pair of meromorphic functions $g(u)$, $h(u)$ of order of growth $\leq\rho$, such that $x=g(u)$, $f(x)=h(u)$ in the neighborhood of $x=\zeta$, such that $g^{-1}(\zeta)'\neq\infty$. Then, for sufficiently large $D \geq D_2(f,m,\varepsilon,\zeta)$, and $c_5=c_5(f,\zeta)>0$, $|c_r|<c_5^r\cdot m^{-(r/\rho)}$.*

Proof. Let $g(u)=g_1(u)/\sigma(u)$, $h(u)=h_1(u)/\sigma(u)$, where $g_1(u)$, $h_1(u)$, $\sigma(u)$ are entire functions it C_u of order of growth $\leq\rho$, such that $x=g(u), f(x) =h(u)$ near $x=\zeta$, $\zeta=g(u_0)$ and $\sigma(u_0)\neq 0$, $g'(u_0)\neq 0$. We put

$$F(u)\overset{\text{def}}{=}\sigma(u)^{D+m-1}R(g(u))=\sum_{i=0}^{m-1}\sigma(u)^{m-1-i}h_1(u)^i\cdot P_i((g_1(u)/\sigma(u))\sigma(u)^D,$$

so that $F(u)$ is an entire function with the following upper bound on a circle $C^T: |u-u_0|=T>0: |F|_T\leq mD\cdot c_2^{mD}\cdot\exp\{\alpha(D+m+1)T^\rho\}$ for $\alpha=\alpha(g_1, h_1, \sigma)>0$.

We can apply now Cauchy theorem to an entire function $F(u)$:

$$(5)\qquad \left|\frac{F^{(r)}(u_0)}{r!}\right|=\left|\frac{1}{2\pi i}\int_{C_T}\frac{F(\zeta)d\zeta}{(\zeta-u_0)^{r+1}}\right|\leq mD\cdot c_2^{mD}\cdot\exp\{A(D+m+1)\cdot T^\rho\}T^{-r}.$$

Here $R^{(r)}(x)|_{x=\zeta}=c_r\cdot r!\neq 0$ and $R^{(n)}(\zeta)=0$ for all $0\leq n\leq r$. Thus

$$F^{(r)}(u)|_{u=u_0}=\sigma(u_0)^{D+m+1}\cdot R^{(r)}(g(u))|_{u=u_0}\cdot g'(u_0)^r$$

and hence,

$$(6)\qquad c_r=(F^{(r)}(u_0)/r!)\cdot\sigma(u_0)^{-(D+m+1)}\cdot g'(u_0)^{-r}.$$

Let us put $T=(r/D)^{1/\rho}$ in (5). Then we get from (5)–(6) $|c_r|<c_2^{mD}\cdot c_6^{D+m}\cdot c_7^r\cdot(r/D)^{-(r/\rho)}$, where $c_6=c_6(u_0)>0$, $c_7=c_7(g, h, u_0)>0$ and $D\geq D_3(\mathrm{m})$. According to Lemma 2, $r\geq mD-[\varepsilon mD]$. Thus $|c_r|\leq c_5^r\cdot m^{-(r/\rho)}$ with $c_5=c_5(f,\zeta)>0$.

As we remarked above, c_r is an algebraic number from K of size at most c_8^r and the denominator at most c_8^r, $c_8=c_8(f,\zeta,\varepsilon)>0$.

Since $c_r\neq 0$, the Norm-product-den $(c_r)\prod_\sigma c_r^{(\sigma)}$ is a nonzero rational integer, where $c_r^{(\sigma)}$ are numbers algebraically conjugate to c_r. Hence $|\mathrm{den}(c_r)\cdot\prod_\sigma c_r^{(\sigma)}|\geq 1$, and using the bound of Lemma 3, we obtain

$$c_5^r\cdot m^{-(r/\rho)}\cdot|\mathrm{den}(c_r)\cdot\prod_{\sigma\neq 1}c_r^{(\sigma)}|\geq 1,\quad\text{or}\quad c_5^r\cdot c_8^{d\cdot r}\geq m^{(r/\rho)},$$

where $d=[K:Q]$. The last inequality is clearly impossible, whenever m is sufficiently large: $m\geq m_0(f, K,\zeta,\varepsilon)$. Hence, if $f(x)$ is a solution of a linear differential equation satisfying assumptions of the Grothendieck conjecture, and $f(x)$ has an algebraic Taylor expansion in the neighborhood of an algebraic point $x=\zeta$, where x and $f(x)$ can be uniformized by meromorphic functions, then $f(x)$ is an algebraic function, cf [5].

§3. Following Dwork's discussion in ([4], §6) we determine now all cases of global nilpotence of Lamé equations. Simultaneously we prove the Grothendieck conjecture for this class of equations. The Lamé equation has the form

$$(7)\qquad P(x)(d^2f/dx^2)+(1/2)P'(x)(df/dx)-\{n(n+1)x+B\}f=0,$$

where $P(x)=4x^3-g_2x-g_3=4(x-e_1)(x-e_2)(x-e_3)\in\bar{Q}[x]$, n is a nonnegative integer and $B\in\bar{Q}$. According to ([6], §23.7), Lamé equation always has two solutions f_+ and f_- such that $f_+\cdot f_-=Q(x, B)$, where $Q(x, B)$ is a polynomial from $\bar{Q}[x, B]$ of degree n. According to [4], [6] there are two possibilities: (i) when f_+/f_- is a constant and, (ii) when f_+, f_- are linearly independent over C. In case (i), B is equal to one of $2n+1$ characteristic values B_n^m $(1\leq m\leq 2n+1)$ of Lamé equation [6] (called in physics ends of lacunae of the spectrum of Lamé equation in the transcendental form, see below). Each of the numbers B_n^m is an algebraic number and one of the solutions of (7) with $B=B_n^m$ is an algebraic function, while there is a non-algebraic solution as well: $m=1, \cdots, 2n+1$. Hence in the case (i), the equation (7) is globally nilpotent (cf. [4], 6.7.1). In the case (ii) as it is shown in ([4], 6.7.2) the global nilpotence of the Lamé equation (7) implies that p-curvature is zero, $\Psi_p=0$, for almost all p.

We use now the transcendental form of Lamé equation (7) and useful remarks from our paper [7]. Let $\wp(u)$ be the Weierstrass elliptic function corresponding to $P(x)$: $\wp'(u)=P(\wp(u))$. Then, under the change of variables, $x=\wp(u)$:

$$(d^2f/du^2)=\{n(n+1)\wp(u)+B\}f. \tag{8}$$

The two solutions f_+, f_- mentioned above have the form

$$f_\pm=\{\textstyle\prod_{i=1}^n(\sigma(a_i\pm u)/\sigma(u)\sigma(a_i))\}\cdot\exp\{\mp u\sum_{i=1}^n\zeta(a_i)\},$$

with the following system of equations on a_i:

$$(2n-1)\textstyle\sum_{i=1}^n\wp(a_i)=B, \qquad \sum_{j=1,j\neq i}^n(\wp'(a_i)+\wp'(a_j))/(\wp(a_i)-\wp(a_j))=0$$

for all $i=1, \cdots, n$. Here $\sigma(u)$ is a Weierstrass' σ-function,

$$\wp(u)=(d^2/du^2)\log\sigma(u),$$

and $\sigma(u)$ is an entire function of order of growth 2. In particular, *any* solution $f=f(u)$ of (8) is a meromorphic function in u of order of growth 2.

Moreover, for $B\neq B_n^m$ $(1\leq m\leq 2n+1)$, two linearly independent solutions of (8) can be expressed in the following form: $f=\sum_{j=0}^{n-1}b_j(d^j/du^j)G(u)$, where $G(u)=(\sigma(u+a)/\sigma(u)\sigma(a))\times\exp\{\rho-\zeta(a)u\}$ and $b_0, \cdots, b_{n-1,\rho}$, $\wp(a)$ are determined algebraically in terms of B and g_2, g_3 [7]. Hence in each of the cases (i) or (ii), assuming the conditions of the Grothendieck conjecture — that $\Psi_p=0$ for almost all p — we deduce as a corollary of Lemmas 2, 3 proved above, that all solutions of (7) are algebraic functions.

Theorem 1. *For an integer* $n\geq 0$ *the Lamé equation* (7) *never satisfies the assumptions of the Grothendieck conjecture*, i.e. $\Psi_p\neq 0$ *for infinitely many* p. *There are* $2n+1$ (*algebraic*) *values of* B, *namely* B_n^m $(1\leq m\leq 2n+1)$, *for which the equation* (7) *is globally nilpotent. For all other values of* B *the equation* (7) *is not globally nilpotent.*

All results of § 2 can be generalized to the case of functions in n variables. This way we obtain a solution to the Grothendieck conjecture [1] for rank one equations over arbitrary algebraic curves (with meromorphic parametrizations given by ratios of θ-functions corresponding to these curves).

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