15. *Diffeomorphism Types of Elliptic Surfaces*

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§ 1. **Statement of results.** The purpose of this note is to announce some results concerning diffeomorphism types of Kodaira's elliptic surfaces [2]. Elliptic surfaces we consider here will satisfy the following conditions:

1) No fiber contains an exceptional curve of the first kind;
2) at least one singular fiber exists;
3) there are no multiple singular fibers.

Theorem 1. *Let* $\Phi_i: M_i \to B_i$ $(i=1,2)$ *be elliptic surfaces satisfying the conditions* 1), 2), 3). *Then there exists an orientation preserving diffeomorphism* $f: M_1 \to M_2$ *if and only if* $g(B_1)=g(B_2)$ *and* $e(M_1)=e(M_2)$, *where* $g(B)$ *and* $e(M)$ *denote the genus of the base curve* B *and the Euler number of the total space* M, *respectively.*

This result extends Kas' theorem [1] which deals with the case $g(B_i)=0$. See also Moishezon [7].

If an elliptic surface $\Phi: M \to B$ satisfies the conditions 1), 2), 3), then by deforming the projection map Φ if necessary, we may (and will) assume that all the singular fibers are of type I_1 ([1], [7]). Let $x_1, x_2, \cdots, x_n$ be the singular loci. We choose a base point $x_0 \in B-\{x_1, \cdots, x_n\}$ and a basis (e_1, e_2) of $H_1(\Phi^{-1}(x_0); Z)$. Then the monodromy representation

$$\rho: \pi_1(B-\{x_1, \cdots, x_n\}, x_0) \longrightarrow SL(2, Z)$$

is well-defined.

We draw loops $L_1, M_1, \cdots, L_g, M_g, g_1, g_2, \cdots, g_n$ on $B-\{x_1, \cdots, x_n\}$ as shown in Fig. 1 (in case $g=g(B)=2$), g_i being a loop based at x_0 which goes round x_i once.

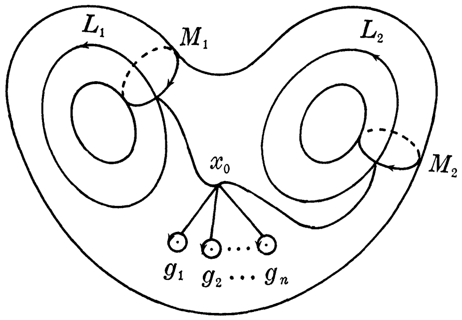

Fig. 1

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Obviously there is a relation in $\pi_1(B-\{x_1, \cdots, x_n\}, x_0)$:

$$[L_1, M_1][L_2, M_2]\cdots[L_g, M_g]=g_1g_2\cdots g_n$$

where $[X, Y]$ denotes the commutator $X^{-1}Y^{-1}XY$.

Theorem 2. *There exists an orientation preserving homeomorphism* $h:(B, \{x_1, \cdots, x_n\}, x_0)\to(B, \{x_1, \cdots, x_n\}, x_0)$ *such that*

$$\rho h(L_1)=\rho h(M_1)=\cdots=\rho h(L_g)=\rho h(M_g)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

This theorem was conjectured by Mandelbaum and Harper [3], and independently by the author [5] (in a weak form). Mandelbaum and Harper announce the theorem for the case $g(B)=1$ and claim that an indirect algebraic-geometric argument exists provided $e(M)>3g(B)$, [3]. Our method is similar to theirs in spirit, but inferring from the rough sketch of their proof in case $g(B)=1$, the details seem considerably different.

Theorem 2, together with Moishezon's normalizing theorem of local monodromies [7, p. 180], gives the following corollary. Theorem 1 follows from this.

Corollary 2.1. *There exists an orientation preserving homeomorphism* $h:(B, \{x_1, \cdots, x_n\}, x_0)\to(B, \{x_1, \cdots, x_n\}, x_0)$ *such that*

1) $\rho h(L_1)=\rho h(M_1)=\cdots=\rho h(L_g)=\rho h(M_g)=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.

2) $\rho h(g_{2i-1})=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$, $\rho h(g_{2i})=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$, $i=1, \cdots, n/2$.

§ 2. Operations. Let $\mathcal{M}$ denote the mapping class group of all the orientation preserving homeomorphisms

$$h:(B, \{x_1, \cdots, x_n\}, x_0)\longrightarrow(B, \{x_1, \cdots, x_n\}, x_0).$$

The set $\mathcal{S}$ of all the configurations of loops that look like Fig. 1 is parametrized by $\mathcal{M}$.

To be precise, we fix a particular configuration $C=(L_1, M_1, \cdots, L_g, M_g; g_1, \cdots, g_n)$, then $\mathcal{S}=\{hC\mid h\in\mathcal{M}\}$.

Define a *right* operation of $\mathcal{M}$ on $\mathcal{S}$ by the rule $\mathcal{S}\times\mathcal{M}\ni(hC, h')\mapsto hh'C\in\mathcal{S}$.

Now we list up the basic operations we use.

Local braids $\varepsilon_1, \varepsilon_1^{-1}, \cdots, \varepsilon_{n-1}, \varepsilon_{n-1}^{-1}$, (cf. [7]): The operations $\varepsilon_i, \varepsilon_i^{-1}$ do not change $(L_1, M_1, \cdots, L_g, M_g)$. ε_i changes $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_n)$ into $(g_1, \cdots, g_ig_{i+1}g_i^{-1}, g_i, \cdots, g_n)$ and ε_i^{-1} $(g_1, \cdots, g_i, g_{i+1}, \cdots, g_n)$ into $(g_1, \cdots, g_{i+1}, g_{i+1}^{-1}g_ig_{i+1}, \cdots, g_n)$. These operations are realized by homeomorphisms which correspond to Artin's braids of n-strings.

Global braids $\mathcal{L}_i, \mathcal{M}_i, \mathcal{L}'_i, \mathcal{M}'_i$ $(i=1, 2, \cdots, g)$: These operate on $\mathcal{S}$ as follows ($\tilde{g}_1$ being a certain conjugate of g_1.):

$\mathcal{L}_i$: $(L_1, M_1, \cdots, L_i, M_i, \cdots, L_g, M_g; g_1, g_2, \cdots, g_n)$
$\to(L_1, M_1, \cdots, \bar{L}_i, M_i, \cdots, L_g, M_g; g_2, \cdots, g_n, \tilde{g}_1)$
where $\bar{L}_i=L_i[M_{i-1}, L_{i-1}]\cdots[M_1, L_1]g_1[L_1, M_1]\cdots[L_{i-1}, M_{i-1}]$,

$\mathcal{M}_i$: $(L_1, M_1, \cdots, L_i, M_i, \cdots, L_g, M_g; g_1, g_2, \cdots, g_n)$
$\to(L_1, M_1, \cdots, L_i, \bar{M}_i, \cdots, L_g, M_g; g_2, \cdots, g_n, \tilde{g}_1)$

where $\bar{M}_i = M_i L_i [M_{i-1}, L_{i-1}] \cdots [M_1, L_1] g_1 [L_1, M_1] \cdots [L_{i-1}, M_{i-1}] L_i^{-1}$,

$\mathcal{L}'_i$: $(L_1, M_1, \cdots, L_i, M_i, \cdots, L_g, M_g ; g_1, g_2, \cdots, g_n)$
$\to (L_1, M_1, \cdots, \bar{L}_i, M_i, \cdots, L_g, M_g ; \tilde{g}_1, g_2, \cdots, g_n)$
where $\bar{L}_i = M_i L_i [M_{i-1}, L_{i-1}] \cdots [M_1, L_1] g_1^{-1} [L_1, M_1] \cdots [L_{i-1}, M_{i-1}] L_i^{-1} M_i^{-1} L_i$,

$\mathcal{M}'_i$: $(L_1, M_1, \cdots, L_i, M_i, \cdots, L_g, M_g ; g_1, g_2, \cdots, g_n)$
$\to (L_1, M_1, \cdots, L_i, \bar{M}_i, \cdots, L_g, M_g ; \tilde{g}_1, g_2, \cdots g_n)$
where $\bar{M}_i = L_i^{-1} M_i L_i [M_{i-1}, L_{i-1}] \cdots [M_1, L_1] g_1^{-1} [L_1, M_1] \cdots [L_{i-1}, M_{i-1}][L_i, M_i]$.

The operations $\mathcal{L}_i$, $\mathcal{M}_i$, $\mathcal{L}'_i$, $\mathcal{M}'_i$ are realized by homeomorphisms which correspond to "global braids" of n-strings in $B-\{x_0\}$. For example $\mathcal{L}_2$ is realized as follows (Fig. 2).

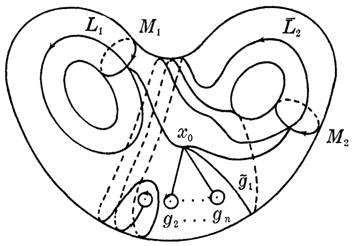

Fig. 2

Dehn twists $D(L_i)$, $D(L_i^{-1})$, $D(M_i)$, $D(M_i^{-1})$ $(i=1, 2, \cdots, g)$: These are the Dehn twists along the loops L_i, M_i, and their inverses.

Theorem 2 is proved by an inductive argument. Let π: $SL(2, \boldsymbol{Z}) \to PSL(2, \boldsymbol{Z})$ be the projection onto the modular group and let a, b denote the images of $\begin{pmatrix} 0, & -1 \\ 1, & 1 \end{pmatrix}$, $\begin{pmatrix} 1, & 2 \\ -1, & -1 \end{pmatrix}$ under π, respectively. Then $PSL(2, \boldsymbol{Z})$ has the presentation $\langle a, b \mid a^3 = 1 = b^2 \rangle$. For an element g of $PSL(2, \boldsymbol{Z})$, the *length* $l(g)$ is defined to be that of the reduced word in a, a^2, b representing g. For example, $l(a^2ba) = 3$. We set $l(1) = 0$.

To each element hC of $\mathcal{S}$, we attach three non-negative integers $\Lambda(hC)$, $\beta(hC)$, $\lambda(hC)$:

$\Lambda(hC) := l(\pi\rho h(L_1)) + l(\pi\rho h(M_1)) + \cdots + l(\pi\rho h(L_g)) + l(\pi\rho h(M_g))$,

$\beta(hC) :=$ the number of $\pi\rho h(L_i)$'s and $\pi\rho h(M_i)$'s which equal to b,

$\lambda(hC) := \sum_{i=1}^{n} l(\pi\rho h(g_i))$.

The proof of Theorem 2 proceeds by induction on the lexicographic order of the triple $(\Lambda, \beta, \lambda)$. In fact, it is proved that if $\Lambda(hC) > 0$, one can find a finite sequence of operations from $\{\varepsilon_1, \varepsilon_1^{-1}, \cdots, \varepsilon_{n-1}, \varepsilon_{n-1}^{-1}, \mathcal{L}_i, \mathcal{M}_i, \mathcal{L}'_i, \mathcal{M}'_i$ $(i=1, \cdots, g)$, $D(L_i)$, $D(L_i^{-1})$, $D(M_i)$, $D(M_i^{-1})$ $(i=1, \cdots, g)\}$ which reduces the order of $(\Lambda(hC), \beta(hC), \lambda(hC))$. At the final stage we must lift the result in $PSL(2, \boldsymbol{Z})$ to that in $SL(2, \boldsymbol{Z})$. This is done using Moishezon's normalizing theorem of local monodromies, [7].

Details will appear elsewhere.

The results of this note can be generalized to obtain a diffeomorphism classification of total spaces of torus fibrations ([4], [6]) over closed oriented surfaces with the simplest singular fibers (I_1^+, I_1^-).

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