14. On Z-Valued Additive Functions on Module Category

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Introduction. Throughout this note, R means a commutative ring with identity and $\mathcal{C}(R)$ the category of finitely generated unitary R-modules. Let S be an additive system, i.e. an algebraic system in which addition is defined. A function L from $\mathcal{C}(R)$ to S will be called an *S-valued additive function over R*, if for any exact sequence $0\to M'\to M\to M''\to 0$ in $\mathcal{C}(R)$, the relation $L(M)=L(M')+L(M'')$ holds. When we want to emphasize that L is over R, we shall write L_R instead of L. In [6], [9] such functions are studied for the case $S=\boldsymbol{R}^+\cup\{\infty\}$. In this note, we study exclusively the case $S=\boldsymbol{Z}$. So we shall simply write "additive functions" for $\boldsymbol{Z}$-valued additive functions. Some arguments in [6], [9] are, however, valid also in our case.

1. Extension of additive functions.

Theorem 1.1. *Let R be noetherian, $\mathfrak{a}$ an ideal of R and n a natural number. Put $A=R/\mathfrak{a}$, $B=R/\mathfrak{a}^n$. Then any additive function L_A over A can be extended to B,* i.e. *there is an additive function L_B over B, such that $L_A(M)=L_B(M)$ for any A-module M.*

Proof. We write the proof of the case $n=2$, since the general case follows easily by induction. So we put $B=R/\mathfrak{a}^2$. Let N be a B-module. We have an exact sequence

$$0\longrightarrow \mathfrak{a}N\longrightarrow N\longrightarrow N/\mathfrak{a}N\longrightarrow 0.$$

Since $\mathfrak{a}(\mathfrak{a}N)=0$ and $\mathfrak{a}(N/\mathfrak{a}N)=0$, $L_A(\mathfrak{a}N)$ and $L_A(N/\mathfrak{a}N)$ are defined. Put $L_B(N)=L_A(\mathfrak{a}N)+L_A(N/\mathfrak{a}N)$. If there is another exact sequence of B-modules

$$0\longrightarrow N_1\longrightarrow N\longrightarrow N_2\longrightarrow 0$$

such that $\mathfrak{a}N_1=\mathfrak{a}N_2=0$, then we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{a}N & \longrightarrow & N & \longrightarrow & N/\mathfrak{a}N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \| & & \downarrow{\scriptstyle\varphi} & & \\ 0 & \longrightarrow & N_1 & \longrightarrow & N & \longrightarrow & N_2 & \longrightarrow & 0 \end{array}$$

From this diagram, we have $\operatorname{Ker}\varphi\simeq\operatorname{Coker}\psi$. Since N_1 and $N/\mathfrak{a}N$ are A-modules, $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\psi$ are also A-modules. Thus we have

$$L_A(\operatorname{Ker}\varphi)=L_A(N/\mathfrak{a}N)-L_A(N_2)=L_A(\operatorname{Coker}\psi)=L_A(N_1)-L_A(\mathfrak{a}N),$$

and hence

$$L_A(N/\mathfrak{a}N)+L_A(\mathfrak{a}N)=L_A(N_1)+L_A(N_2).$$

Now we prove the additivity of L_B. Let there be given an exact sequence of B-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

From this exact sequence, we obtain a commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_1 & \longrightarrow & \mathfrak{a}M_2 & \xrightarrow{\varphi} & \mathfrak{a}M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_2 & \longrightarrow & M_2/\mathfrak{a}M_2 & \xrightarrow{\psi} & M_3/\mathfrak{a}M_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $K_1 = \operatorname{Ker}\varphi$, $K_2 = \operatorname{Ker}\psi$ are A-modules. Hence we have

$$\begin{aligned}
L_B(M_2) &= L_A(\mathfrak{a}M_2) + L_A(M_2/\mathfrak{a}M_2) \\
&= L_A(K_1) + L_A(\mathfrak{a}M_3) + L_A(K_2) + L_A(M_3/\mathfrak{a}M_3) \\
&= L_B(M_1) + L_B(M_3).
\end{aligned}$$

Theorem 1.2. *Let R be noetherian. Then any additive function L_R over R can be extended to an additive function $L_{R[x]}$ over the polynomial ring $R[x]$ over R with one valuable x,* i.e. *there exists an additive function $L_{R[x]}$ over $R[x]$ such that*

$$L_R(M) = L_{R[x]}(M \otimes_R R[x])$$

for any $M \in \mathcal{C}(R)$.

Proof. The following definition of $L_{R[x]}$ is as in [4, p. 407]. Let K_x be a Koszul complex

$$\cdots \longrightarrow 0 \longrightarrow R[x] \xrightarrow{x} R[x] \longrightarrow 0 \longrightarrow \cdots.$$

Put $K(x, N) = K_x \otimes_{R[x]} N$ for any $N \in \mathcal{C}(R[x])$. We define, for any $N \in \mathcal{C}(R[x])$, $L_{R[x]}(N) = \chi(H(K(x, N)))$ where the right hand side is the Euler characteristic of the Koszul complex $K(x, N)$ [cf. 8], i.e.

$$L_{R[x]}(N) = L_R(N/xN) - L_R((0:x)_N).$$

Then $L_{R[x]}$ is an additive function over $R[x]$. If $M \in \mathcal{C}(R)$, then $L_{R[x]}(M \otimes_R R[x]) = L_R(M)$ since $(0:x)$ in $M \otimes_R R[x]$ is zero.

Note that L_R can be extended to $L_{R[x_1, \ldots, x_n]}$ by induction on n.

2. Trivial additive functions. Let R be an integral domain and c any integer. The function $c \operatorname{rank}_R M$ is obviously an additive function over R. Additive function of this type will be called *trivial.*

Theorem 2.1. *If R is a regular local ring, any additive function over R is trivial and moreover $L(M) = L(R) \operatorname{rank}_R M$ for $M \in \mathcal{C}(R)$.*

To prove this, we use the following lemma.

Lemma 2.2. *If $M \in \mathcal{C}(R)$ has a finite free resolution and if there is a non zero-devisor s of R such that $sM = 0$, then $L(M) = 0$ for any additive function L over R.*

Proof. Let

$$0 \longrightarrow F_t \longrightarrow F_{t-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a free resolution of M. From this exact sequence we have

$$L(M)=L(F_0)-L(F_1)+\cdots+(-1)^t L(F_t)$$

by the additivity of L. Let S be the set of all non zero-divisors of R. The hypothesis $sM=0$ implies that the sequence

$$0 \longrightarrow S^{-1}F_t \longrightarrow \cdots \longrightarrow S^{-1}F_0 \longrightarrow 0$$

is exact. Hence $\sum_{i=0}^{t}(-1)^i \operatorname{rank} F_i=0$. Since $L(F_i)=L(R) \operatorname{rank} F_i$, we have $L(M)=0$.

Since, for an R-sequence $x_1, \cdots, x_t$, the Koszul complex $K(R_{x:1,\dots,t})$ is a free resolution of $R/(x_1, \cdots, x_t)R$, we have:

Corollary 2.3. *Let $x_1, \cdots, x_t$ be an R-sequence, $t \geqq 1$. Then we have $L(R/(x_1, \cdots, x_t)R)=0$ for any additive function L over R.*

Proof of Theorem 2.1. Let N_i be submodules of M such that

$$M=N_0 \supset N_1 \supset \cdots \supset N_t=0 \quad \text{and}$$
$$N_i/N_{i+1} \simeq R/P_i$$

where $P_i \in \operatorname{Spec} R$ for all i, $0 \leqq i \leqq t-1$. Then we have $L(M)=\sum_{i=0}^{t-1} L(R/P_i)$ by the additivity of L. If $P_i \neq 0$, then we have $L(R/P_i)=0$ by Lemma 2.2. Let m be the number of modules N_i/N_{i+1} with the property that $N_i/N_{i+1} \simeq R$ in the system $\{N_i/N_{i+1}\}_{i=0,\dots,t-1}$. Then $m=\operatorname{rank}_R M$ since $\operatorname{rank}_R M$ is the dimension of $S^{-1}M$ over $K=S^{-1}R$, where $S=R-\{0\}$.

Remark. This result is proved by the fact that the Grothendieck group $K_0(R)$ is isomorphic to $\boldsymbol{Z}$.

Theorem 2.4. *Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_n)$ be a semi-local ring of dimension* 2. *If R is a unique factorization domain, any additive function over R is trivial and $L(M)=L(R) \operatorname{rank}_R M$ for $M \in \mathcal{C}(R)$.*

For the proof, we use the following lemma.

Lemma 2.5. *Let $(R, \mathfrak{m}_1, \cdots, \mathfrak{m}_n)$ be a semi-local ring of* $\dim R \geqq 1$. *Then we have $L(R/\mathfrak{m}_i)=0$ for any $\mathfrak{m}_i$ with* $\operatorname{ht} \mathfrak{m}_i \geqq 1$ *and for any additive function L over R.*

Proof. Let $P \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{m}/P=1$ and $x \in \mathfrak{m}_i-(\bigcup_{j \neq i} \mathfrak{m}_j \cup P)$. We have an exact sequence

$$0 \longrightarrow R/P \xrightarrow{x} R/P \longrightarrow R/(P, x) \longrightarrow 0.$$

From this exact sequence, we have $L(R/(P, x))=0$. Since (P, x) is $\mathfrak{m}_i$-primary, there are submodules N_j of $R/(P, x)$ such that

$$R/(P, x)=N_0 \supset N_1 \supset \cdots \supset N_t=0$$

with $N_j/N_{j+1} \simeq R/\mathfrak{m}_i$ for all j, $0 \leqq j \leqq t-1$. This implies that $0=L(R/(p, x)) =tL(R/\mathfrak{m}_i)$, hence $L(R/\mathfrak{m}_i)=0$.

Proof of Theorem 2.4. Let P be a prime ideal of height 1. Since R is a U.F.D., P is principal, say $P=(x)$. The exact sequence

$$0 \longrightarrow R \xrightarrow{x} R \longrightarrow R/P \longrightarrow 0$$

implies $L(R/P)=0$, for any additive function L. Lemma 2.5 and this fact imply the desired result in the same way as in [6].

Proposition 2.6. *Any additive function over a polynomial ring $R=k[x_1,\cdots,x_n]$ over a field k is trivial.*

The proof is the same as in Theorem 2.1.

The following result was suggested by K. Hirata.

Theorem 2.7. *Let R be a noetherian integral domain, then the additive function $L_{R[x]}$ constructed in Theorem 1.2 is trivial if L_R is trivial.*

Proof. It suffices to prove that $L_A(A/P)=0$ for any non-zero prime ideal P of A where $A=R[x]$. Let $P\in \operatorname{Spec} A$ and $P\neq 0$. If $P\ni x$, put $M=A/P$. Then we have $xM=0$, and hence $M/xM=M$ and $(0:x)_M=M$. This implies $L_A(M)=0$. If $P\not\ni x$, put $M=A/P$. Then we have $M/xM=A/(P,x)$ and $(0:x)_M=0$. If we put $\mathfrak{a}=R\cap(P,x)$, then $\mathfrak{a}\neq 0$. Since $A/(P,x)$ is isomorphic to $R/\mathfrak{a}$ as R-module, we have $L_A(M)=L_R(R/\mathfrak{a})=0$.

3. Non-trivial additive functions. We cite the following result of S. Kondo (unpublished).

Theorem 3.1. *Let R be a Dedekind domain, $\tilde{K}_0$ the reduced group of the Grothendieck group of $C(R)$. Then the following conditions* (i), (ii) *are equivalent.*

(i) *Any additive function over R is trivial.*

(ii) $\operatorname{Hom}(\tilde{K}_0, \boldsymbol{Z})=0$.

Now it is known that $\tilde{K}_0$ is isomorphic to the ideal class group of R, and that there exists R such that this group is isomorphic to any given abelian group. This means that (ii) does not hold in general, i.e. non-trivial additive functions exist over certain Dedekind domains. Theorem 3.1 holds for any integral domain R but it is unknown to the author whether non-trivial additive function exists in other case.

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