81. On the Existence and Uniqueness of SDE Describing an n-particle System Interacting via a Singular Potential

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1. Introduction. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a filtered probability space and let $[B^1(t), \cdots, B^n(t)]$ be an $\mathcal{F}_t-BM_0^{2nd}$ ($B^i(t)$ is $\mathbf{R}^{2d}$-valued), where BM_x^r denotes an r-dim. Brownian motion starting from $x \in \mathbf{R}^r$. Let $\mathcal{N}$ denote the set of points $(z^1, \cdots, z^n) \in \mathbf{R}^{2nd}$ such that $z^i = z^j$ for some $i \neq j$ and $z^\perp$ denote $(y, -x)$ for $z=(x, y) \in \mathbf{R}^{2d}$. We consider the following stochastic differential equation (abbreviated: *SDE*) *describing an interacting n-particle system* in $\mathbf{R}^{2d}$ starting from $(z^1, \cdots, z^n) \notin \mathcal{N}$:

$$(1)\qquad \begin{aligned} &dZ^i(t) = dB^i(t) + \textstyle\sum_{j:j\neq i} \gamma_j \nabla^\perp H(Z^i(t) - Z^j(t))dt \qquad i=1, \cdots, n,\\ &Z^i(0) = z^i \qquad i=1, \cdots, n,\end{aligned}$$

in which,

$$\gamma_i \in \mathbf{R}^1,\ \neq 0 \qquad i=1, \cdots, n,$$
$$H(z) = g(|z|), \quad (\nabla^\perp H)(z) = (\nabla H(z))^\perp \quad z \in \mathbf{R}^{2d},\ \neq 0,$$

where $g \in C^2(0, \infty)$ and $\nabla H = (\partial H/\partial z_1, \cdots, \partial H/\partial z_{2d}) \in \mathbf{R}^{2d}$. For a typical example, if we set $g(r) = -(1/2\pi)\log r$ and $d=1$, then the above system of *SDE* describes a dynamics of n vortices in incompressible and viscous fluid in $\mathbf{R}^2$, where the constants γ_i denote the vorticity of the i-th vortex ([1], [3]). Hence we call this the *SDE representing the vortex flow.* (1) is significant in connection with the nonlinear *SDE* in $\mathbf{R}^{2d}$:

$$dZ(t) = dB(t) + \int_{\mathbf{R}^{2d}} \nabla^\perp H(Z(t) - z)\mu_t(dz)dt,$$

where $B(t)$ is a BM_0^{2d} and $\mu_t(dz)$ is the law of $Z(t)$. Particularly the *SDE representing the vortex flow* is related to the *Navier-Stokes equation* ([3]).

The problem we consider is the existence and uniqueness of a solution of (1). In fact H. Osada ([4]) proved that in the vortex flow case, (1) has a unique strong solution, using an estimate of the fundamental solution of a parabolic equation with a *generalized divergence form.* In this paper, under a suitable condition on the singularity of $g(r)$ at $r=0$ and assuming that $\{\gamma_i\}$ has the same sign, we prove the unique existence of a solution for a general (1) including the vortex flow case by a *probabilistic* method, which seems simpler than Osada's. But in Osada's argument, the equi-sign property of $\{\gamma_i\}$ is not necessary.

One can explain intuitively the reason why the equi-sign property of $\{\gamma_i\}$ simplifies the situation: Assuming $g'(r) > 0$, we can see that the drift acts on $\{Z^i, Z^j\}$ as if Z^i and Z^j rotate around $(Z^i + Z^j)/2$ clockwise with intensities $\gamma_j g'(r)$ and $\gamma_i g'(r)$ $(r = |Z^i - Z^j|)$ respectively. This fact prevents

Z^i and Z^j from approaching each other.

2. Result. We are interested in the case: $g'(r)$ is unbounded near $r=0$ because, otherwise, (1) has a unique strong solution. From now on we only consider $g(r)\in C^2(0,\infty)$ satisfying the following conditions:

$$g(0+)=+\infty \quad \text{or} \quad g(0+)=-\infty, \tag{2}$$

$$\begin{aligned} g''+(2d-1)/rg' &\leqq \text{const.} \quad \text{if } g(0+)=+\infty, \\ &\geqq \text{const.} \quad \text{if } g(0+)=-\infty, \end{aligned} \tag{3}$$

$$g'(r)=O(r) \quad \text{as } r\uparrow\infty. \tag{4}$$

It is easy to see that $g\in C^2(0,\infty)$ satisfies (2)–(4) if and only if $g(r)$ is of the form

$$g(r)=c_1+c_2r^2+\int_1^r \rho(s)/s^{2d-1}ds,$$

in which, $c_1, c_2\in \mathbf{R}^1$ and ρ is a C^1 and monotonic function on $(0,\infty)$ such that (i) $\int_{0+}\rho(r)/r^{2d-1}dr=+\infty$ or $-\infty$ accordingly as $\rho\in\uparrow$ or $\rho\in\downarrow$, and (ii) $\rho(r)=O(r^{2d})$ as $r\uparrow\infty$. First of all we note that

$$r^{2d-1}g'(r) \quad \text{is bounded near } r=0. \tag{5}$$

We assume that a starting point $(z^1,\cdots,z^n)$ does not belong to $\mathfrak{N}$ unless otherwise stated. Since $g'(r)$ is divergent at $r=0$, the existence of a solution is not obvious. But (1) is meaningful up to hitting $\mathfrak{N}$, because $\nabla^{\perp}H(z)$ is locally Lipschitz continuous outside $z=0$ and grows in linear order as $|z|\uparrow\infty$ on $|z|\geqq\varepsilon$ for each $\varepsilon>0$, which results from (4). More precisely, for a small enough $\varepsilon>0$ we set

$$\tau_{ij}^{\varepsilon}=\inf\{t\geqq 0: r_{ij}(t)=\varepsilon\} \qquad i\neq j,$$
$$\tau^{\varepsilon}=\min_{i\neq j}\tau_{ij}^{\varepsilon},$$

where $r_{ij}(t)=|Z^i(t)-Z^j(t)|$. Clearly $0<\tau^\varepsilon\leqq+\infty$ and τ^ε is nondecreasing as $\varepsilon\downarrow 0$. If we define $\tau=\lim_{\varepsilon\downarrow 0}\tau^\varepsilon$, then it is easily seen that (1) has a unique solution up to τ.

Theorem. *Suppose that $\gamma_i>0$ for all i or $\gamma_i<0$ for all i. Then for any starting point $(z^1,\cdots,z^n)\notin\mathfrak{N}$, $P(\tau=+\infty)=1$.*

If this theorem is proved, then the unique existence of a solution of (1) is immediate as just mentioned.

3. Proof of Theorem. For the statement of the following lemma, which plays an important role in our arguments, we introduce the following definition. Let σ be an $\mathcal{F}_t$-stopping time for which there exists a sequence of nondecreasing $\mathcal{F}_t$-stopping times $\{\sigma_n\}_{n=1}^\infty$ such that $0<\sigma_n<\sigma$ $n=1,2,\cdots$ and $\sigma_n\uparrow\sigma$ as $n\uparrow\infty$. We say that $[m(t):0\leqq t<\sigma]$ is an *$\mathcal{F}_t$-local martingale up to σ* if $[m(t\wedge\sigma_n):t\geqq 0]$ is an $\mathcal{F}_t$-local martingale for each $n\geqq 1$. Then we can define a quadratic variational process $[\langle m\rangle(t):0\leqq t<\sigma]$ up to σ as follows: $\langle m\rangle(t)=\langle m_n\rangle(t)$ for $0\leqq t\leqq\sigma_n$ and $n\geqq 1$, where $m_n(t)=m(t\wedge\sigma_n)$ $t\geqq 0$. Hence, from our definition $[m^2(t)-\langle m\rangle(t):0\leqq t<\sigma]$ is an $\mathcal{F}_t$-local martingale up to σ.

Lemma 1. *Let $[m(t):0\leqq t<\sigma]$ be an $\mathcal{F}_t$-local martingale up to σ with $m(0)=0$. Then with probability one, either* (i) *or* (ii) *occurs:*

(i) *$m(t)$ has a finite limit as $t\uparrow\sigma$,*

(ii) $\limsup_{t\uparrow\sigma} m(t)=+\infty$, $\liminf_{t\uparrow\sigma} m(t)=-\infty$.

Proof. We set $\varphi_p(t)=\inf\{s:\langle m_p\rangle(s)>t\}$ for $p\geqq 1$. By the same argument as Theorem II-7.2′ in [2], $\tilde{B}(t)=\lim_{p\uparrow\infty} m(\sigma_p\wedge\varphi_p(t))$ exists a.s. and we can regard $\tilde{B}$ as a (1-dim.) Brownian motion up to $\langle m\rangle(\sigma-)$. On the other hand it is easy to see that $\tilde{B}(\langle m\rangle(t))=m(t)$ for $0\leqq t<\sigma$. From this, the lemma is proved immediately.

We assume that the sign of all γ_i is the same. From Itô's formula we have

$$(6)\qquad \textstyle\sum_{i=1}^n \gamma_i|Z^i(t)|^2=\sum_{i=1}^n\gamma_i|z^i|^2+\sum_{i=1}^n 2\gamma_i\displaystyle\int_0^t Z^i\cdot dB^i+(2d)(\textstyle\sum_{i=1}^n\gamma_i)t,$$

$$\begin{aligned}(7)\qquad &\textstyle\sum_{i\neq j}\gamma_i\gamma_j H(Z^i(t)-Z^j(t))\\ &=\textstyle\sum_{i\neq j}\gamma_i\gamma_j H(z^i-z^j)+\sum_{i\neq j}\gamma_i\gamma_j\displaystyle\int_0^t \nabla H(Z^i-Z^j)\cdot d(B^i-B^j)\\ &\quad+\textstyle\sum_{i\neq j}\gamma_i\gamma_j\displaystyle\int_0^t g''(r)+(2d-1)/r g'(r)|_{r=|Z^i-Z^j|}ds,\end{aligned}$$

for $0\leqq t<\tau$. We choose $\gamma>0$ so that $\gamma<\min_{1\leqq i\leqq n}|\gamma_i|/\sqrt{d}$ and define $\eta_i=\gamma_i/\gamma$ for $i=1,\cdots,n$. Then using Itô's formula, we have

$$\begin{aligned}(8)\qquad &\textstyle\prod_{i\neq j}|Z^i(t)-Z^j(t)|^{\eta_i\eta_j}=\prod_{i\neq j}|z^i-z^j|^{\eta_i\eta_j}\\ &+\textstyle\sum_{i\neq j}\eta_i\eta_j\displaystyle\int_0^t|Z^i-Z^j|^{-2}(\textstyle\prod_{p\neq q}|Z^p-Z^q|^{\eta_p\eta_q})(Z^i-Z^j)\cdot d(B^i-B^j)\\ &+\textstyle\sum_{i\neq j}2\eta_i\eta_j(\eta_i\eta_j+d-1)\displaystyle\int_0^t|Z^i-Z^j|^{-2}(\textstyle\prod_{p\neq q}|Z^p-Z^q|^{\eta_p\eta_q})ds\\ &+\textstyle\sum_{\substack{i<j,k<l\\(i,j)\neq(k,l)}}2\eta_i\eta_j\eta_k\eta_l\displaystyle\int_0^t|Z^i-Z^j|^{-2}|Z^k-Z^l|^{-2}(\textstyle\prod_{p\neq q}|Z^p-Z^q|^{\eta_p\eta_q})\\ &\quad\times(Z^i-Z^j)\cdot(Z^k-Z^l)(\delta_{ik}-\delta_{il}-\delta_{jk}+\delta_{jl})ds\\ &+2/\gamma^2\textstyle\sum_{i\neq j,j\neq k,k\neq i}\gamma_i\gamma_j\gamma_k\displaystyle\int_0^t|Z^i-Z^j|^{-2}|Z^i-Z^k|^{-2\eta_i\eta_k}\\ &\quad\times(\textstyle\prod_{p\neq q}|Z^p-Z^q|^{\eta_p\eta_q})|Z^i-Z^k|^{2\eta_i\eta_k-1}g'(|Z^i-Z^k|)\\ &\quad\times(Z^i-Z^j)\cdot(Z^i-Z^k)^{\perp}ds,\end{aligned}$$

for $0\leqq t<\tau$. Here δ_{ij} denotes Kronecker's delta.

Lemma 2. $\sup_{0\leqq t<\tau}|Z^i(t)|<+\infty$ $i=1,\cdots,n$ *a.s. on* $\{\tau<+\infty\}$.

Proof. Since the left side of (6) is either nonnegative or nonpositive, (ii) in Lemma 1 for a local martingale $\sum_{i=1}^n 2\gamma_i\int_0^t Z^i\cdot dB^i$ up to τ can not happen a.s. on $\{\tau<+\infty\}$. This implies that the left side of (6) has a finite limit as $t\uparrow\tau$ a.s. on $\{\tau<+\infty\}$. Therefore, using the equi-sign property of $\{\gamma_i\}$, we have Lemma 2.

Lemma 3. *As* $t\uparrow\tau$, $\sum_{i\neq j}\gamma_i\gamma_j H(Z^i(t)-Z^j(t))$ *tends to* $+\infty$ *a.s. on* $\{\tau<+\infty\}$ *or* $-\infty$ *a.s. on* $\{\tau<+\infty\}$ *according to* $g(0+)=+\infty$ *or* $g(0+)=-\infty$.

Proof. From (5), Lemma 2 and the fact: $\eta_i\eta_j>d$ for any i and j, we observe that the integrands in the right side of (8) are bounded on $[0,\tau)$ a.s. on $\{\tau<+\infty\}$. Hence we see that the left side of (8) has a finite limit as $t\uparrow\tau$ a.s. on $\{\tau<+\infty\}$. In fact this limit is zero, because the left side of (8) converges to zero as t tends to τ along τ^{ε} when $\tau<+\infty$. This, together with (2) Lemma 2, and the equi-sign property of $\{\gamma_i\}$, implies the assertion of the lemma.

Proof of Theorem. Without loss of generality we may assume $g(0+) = +\infty$. From (3), (7) and the equi-sign property of $\{\gamma_i\}$, we have

(9) $\sum_{i\neq j}\gamma_i\gamma_j H(Z^i(t)-Z^j(t)) \leqq \sum_{i\neq j}\gamma_i\gamma_j H(z^i-z^j)+M(t)+\text{const.}\, t$

for $0\leqq t<\tau$ with a local martingale $M(t)$ up to τ. Applying Lemma 1, we observe that the right side of (9) does not tend to $+\infty$ as $t\uparrow\tau$ a.s. on $\{\tau<+\infty\}$. But this contradicts the conclusion of Lemma 3. Therefore $P(\tau<+\infty)=0$.

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