# 48. Continuum of Ideals in $R(\Phi_2)\otimes_{\max} R'(\Phi_2)$

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Let $\Phi_2$ be the free group on two generators $a$ and $b$. Let $\mathcal{H}=\mathcal{L}^2(\Phi_2)$ be the Hilbert space of all complex valued functions $f(g)$ on $\Phi_2$ such that

$$\sum_{g\in\Phi_2}|f(g)|^2<\infty.$$

For each $g_1\in\Phi_2$ we define the unitary operator $U(g_1)$ on $\mathcal{H}$ given by

$$(U(g_1)f)(g)=f(g_1^{-1}g), \qquad \text{for all } f\in\mathcal{H}.$$

The von Neumann algebra generated by $\{U(g),\ g\in\Phi_2\}$ is denoted by $R(\Phi_2)$. It is known that $R(\Phi_2)$ is a $\mathrm{II}_1$-factor.

The purpose of this paper is to show the existence of continuum of ideals in $R(\Phi_2)\otimes_{\max} R'(\Phi_2)$.

We will use the following universal property of the projective $C^*$-tensor product.

**Lemma 1.** *Given $C^*$-algebras $A_1$, $A_2$ and $B$, if $\pi_1: A_1\to B$ and $\pi_2: A_2\to B$ are homomorphisms with commuting ranges, then there exists a unique homomorphism $\pi$ of the projective $C^*$-tensor product $A_1\otimes_{\max} A_2$ into $B$ such that*

$$\pi(x_1\otimes x_2)=\pi_1(x_1)\pi_2(x_2) \qquad x_1\in A_1,\ x_2\in A_2,$$

*and the image $\pi(A_1\otimes_{\max} A_2)$ is the $C^*$-subalgebra of $B$ generated by $\pi_1(A_1)$ and $\pi_2(A_2)$* (cf. [4, p. 207]).

We denote by $\mathrm{Int}(R(\Phi_2))$ and $\mathrm{Aut}(R(\Phi_2))$ the set of all inner automorphisms and that of all automorphisms of $R(\Phi_2)$ respectively, with the topology of strong pointwise convergence in $R(\Phi_2)$.

**Lemma 2.** $\mathrm{Int}(R(\Phi_2))$ *is closed in* $\mathrm{Aut}(R(\Phi_2))$.

For the proof see [3, Corollory 3.8].

In the following we will use the Connes's characterization of approximately inner automorphisms.

**Lemma 3.** *Let $N$ be a factor of type* $\mathrm{II}_1$ *with separable predual acting in $\mathcal{K}=L^2(N,\tau)$. Then the following conditions are equivalent for $\theta\in\mathrm{Aut}(N)$,*

(a) $\theta\in\overline{\mathrm{Int}(N)}$;

(b) *There exists an automorphism of the $C^*$-algebra generated by $N$ and $N'$ in $\mathcal{K}$ which is $\theta$ on $N$ and identity on $N'$* ([2, p. 89]).

In Lemma 1, if we put $A_1=R(\Phi_2)$, $A_2=R'(\Phi_2)$ and $\pi_1$, $\pi_2$ as identical map, there exists a homomorphism $\eta$ such that

$$R\underset{\max}{\otimes} R'\xrightarrow[\text{onto}]{\eta}C^*(R,R'),$$

$$R\underset{\max}{\otimes} R'/I\cong C^*(R,R')$$

in which $I$ is $\mathrm{Ker}(\eta)$.

For any $\alpha \in \mathrm{Aut}(R(\Phi_2))$, the automorphism $\alpha\otimes Id$ defined on the algebraic tensor product of $R$ and $R'$ can be uniquely extended to $R\otimes_{\max} R'$. It is still denoted by $\alpha\otimes Id$.

**Lemma 4.** *$I$ is a proper ideal of $R\otimes_{\max} R'$.*

*Proof.* If $I$ were $\{0\}$, there would exist an isomorphism $\eta^*$ from $R\otimes_{\max} R'$ to $C^*(R, R')$.

By [1, p. 593, Corollary 2], the outer automorphism of $\Phi_2$ changing two generators can be extended to the outer automorphism of $R(\Phi_2)$.

Choosing $\alpha \overline{\in} \overline{\mathrm{Int}(R(\Phi_2))} = \mathrm{Int}(R(\Phi_2))$, we define

$$\bar{\alpha}(z) = \eta^*(\alpha\otimes Id)\eta^{*-1}(z) \qquad z \in C^*(R, R').$$

Therefore, $\bar{\alpha}(xy) = \alpha(x)y$, for $x \in R$, $y \in R'$.

It follows that $\bar{\alpha}$ is an automorphism on $C^*(R, R')$, which is $\alpha$ on $R$ and identity on $R'$. By use of Lemma 3, $\alpha \in \mathrm{Int}(R(\Phi_2))$. It is a contradiction.

**Lemma 5.** *If $\alpha, \beta \in \mathrm{Aut}(R(\Phi_2))$, $(\alpha\otimes Id)(I) = (\beta\otimes Id)(I)$ if and only if $\alpha^{-1}\beta \in \mathrm{Int}(R(\Phi_2))$.*

*Proof.* If $(\alpha\otimes Id)(I) = I$, we will prove $\alpha \in \mathrm{Int}(R(\Phi_2))$. We put $\eta_1 = \eta(\alpha\otimes Id)$. Then $\eta_1$ is a homomorphism from $R\otimes_{\max} R'$ onto $C^*(R, R')$.

By Lemma 1, if we put $A_1 = R(\Phi_2)$, $A_2 = R'(\Phi_2)$ and $\pi_1 = \alpha$, $\pi_2$ is identical map, $\eta_1$ is the homomorphism such that

$$R \underset{\max}{\otimes} R' \xrightarrow[\text{onto}]{\eta_1} C^*(R, R').$$

We denote $\mathrm{Ker}(\eta_1)$ by $I_\alpha$. It is then clear that $\mathrm{Ker}(\eta_1) = (\alpha^{-1}\otimes Id)(I)$. Then we consider the canonical decomposition of $\eta_1$:

$$R \underset{\max}{\otimes} R' \longrightarrow R \underset{\max}{\otimes} R'/I \xrightarrow{\alpha\odot Id} \eta_1(R \underset{\max}{\otimes} R') = C^*(R, R').$$

Since $(\alpha\otimes Id)(I) = I$, then $I_\alpha = I$.

Therefore, $\alpha\odot Id$ is an automorphism of $C^*(R, R')$, which is $\alpha$ on $R$ and identity on $R'$. By Lemma 3, $\alpha \in \mathrm{Int}(R(\Phi_2))$. If $\alpha \in \mathrm{Int}(R(\Phi_2))$, by Lemma 3, there is an automorphism $\alpha\odot Id \in \mathrm{Aut}(C^*(R, R'))$ which is $\alpha$ on $R$ and identity on $R'$. Since $C^*(R, R') = R\otimes_{\max} R'/I$ we have that $(\alpha\otimes Id)(I) = I$.

**Lemma 6.** *There is a group of outer automorphisms with continuous parameter in* $\mathrm{Aut}(R(\Phi_2))$.

*Proof.* From [1, Theorem 5.2], we have the following situation. Let $\{\lambda_\alpha ; \alpha \in \Phi_2\}$ be a set of complex numbers of absolute value 1 with $\lambda_{\alpha\beta} = \lambda_\alpha\lambda_\beta$ then $S(U_\alpha) = \lambda_\alpha U_\alpha$ defines a spatial automorphism of $R(\Phi_2)$, where $\{U_\alpha ; \alpha \in \Phi_2\}$ is the unitary representation of $\Phi_2$ defined in [1]. The group of all such automorphisms forms a group of outer automorphisms if $(\Phi_2)_0$ is the center of $\Phi_2$, where $(\Phi_2)_0$ denotes the normal subgroup of $\Phi_2$ consisting of all elements in $\Phi_2$ with finite conjugacy classes.

Now, evidently, $(\Phi_2)_0$ is the center of $\Phi_2$, so that there is a group of outer automorphisms of $\Phi_2$ with continuous parameter.

**Theorem.** *There is continuum of ideals in $R\otimes_{\max} R'$.*

*Proof.* By Lemma 6, in $\mathrm{Aut}(R(\Phi_2))$ there is a group of outer automorphisms with continuous parameter which is denoted by $\{\alpha_\lambda\}$, $\lambda \in [a, b]$.

Setting $I_\lambda=(\alpha_\lambda\otimes Id)(I)$, by Lemma 5, it follows that $I_\lambda\neq I_\mu$ for $\lambda\neq\mu$. So $\{I_\lambda\}$ is continuum of ideals in $R\otimes_{\max} R'$.

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