30. *Family of Jacobian Manifolds and Characteristic Classes of Surface Bundles. II*

By Shigeyuki MORITA
Department of Mathematics, College of Arts and Sciences,
University of Tokyo

(Communicated by Kunihiko KODAIRA, M. J. A., April 12, 1985)

1. Introduction. This note is a sequel to our previous papers [2], [3]. There we have investigated cohomological properties of a canonical map, called the Jacobi mapping, from a surface bundle with a cross section to its associated family of Jacobian manifolds and from them we derived new relations among our characteristic classes of surface bundles. The purpose of the present note is to announce new related results. Namely we have obtained still more relations by applying the techniques of [3] to surface bundles *without* cross sections. More precisely in case of a surface bundle with cross section, the structure group of the associated family of Jacobian manifolds was the Siegel modular group $Sp(2g;\boldsymbol{Z})$ which acts on T^{2g} linearly and preserving a prescribed symplectic form ω_0. In the general case we enlarge the structure group to the semi-direct product $T^{2g}\rtimes Sp(2g;\boldsymbol{Z})$. Namely we allow the translations of T^{2g}. The natural action of $T^{2g}\rtimes Sp(2g;\boldsymbol{Z})$ on T^{2g} still preserves the form ω_0. Now we show that for any given surface bundle $\pi: E\to X$, there is a canonical flat T^{2g}-bundle $\pi': J'\to X$ with structure group $T^{2g}\rtimes Sp(2g;\boldsymbol{Z})$ and a natural fibre preserving map $j': E\to J'$ such that the restriction of j' to each fibre induces an isomorphism on the first homology (see Corollary 2). This should be considered as the topological version of Earle's embedding theorem [1] which states that any holomorphic family of compact Riemann surfaces over a complex manifold can be embedded in a certain associated family of Jacobian varieties in an essentially unique way. Earle's family of Jacobian varieties is not the same as the one defined in [3] in general. In fact it may not have any cross section. Moreover even if a surface bundle $\pi: E\to X$ admits a cross section, the flat T^{2g}-bundle $\pi': J'\to X$ above is not in general isomorphic to the previously defined bundle $\pi: J\to X$ ([3]) as *flat* bundles (see § 3). Using this fact we can obtain strong relations among our characteristic classes (see Corollary 6).

2. Topological version of Earle's embedding theorem. Henceforth we use the terminologies of [2], [3] freely. In particular $\mathcal{M}_g$ and $\mathcal{M}_{g,*}$ respectively are the mapping class groups of the closed oriented surface Σ_g of genus $g\geqq 2$ and Σ_g relative to the base point. As in § 6 of [3], we define a crossed homomorphism

$$f_0: \mathcal{M}_{g,*}\times H_1(\Sigma_g)\longrightarrow \boldsymbol{Z}$$

as follows. Let $\Sigma_g^0=\Sigma_g-\mathring{D}^2$ and let $\mathcal{M}_{g,1}$ be the mapping class group of Σ_g^0. $\pi_1(\Sigma_g^0)$ is a free group on $2g$ generators $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g$. For each element $\gamma \in \pi_1(\Sigma_g^0)$, we denote $[\gamma] \in H_1(\Sigma_g^0)\cong H_1(\Sigma_g)$ for the homology class of γ and also we denote $[\gamma]\cdot[\gamma']$ for the intersection number of $[\gamma]$ and $[\gamma']$ $(\gamma, \gamma' \in \pi_1(\Sigma_g^0))$. Any element $\gamma \in \pi_1(\Sigma_g^0)$ can be uniquely expressed as

$$\gamma=\gamma_1\cdots\gamma_n$$

where γ_i is $\alpha_j^{\pm}$ or $\beta_j^{\pm}$ for some j. We define

$$d(\gamma)=\sum_{i=1}^{n-1}[\gamma_i]\cdot[\gamma_{i+1}\cdots\gamma_n].$$

For each element $\phi \in \mathcal{M}_{g,*}$ and $x \in H_1(\Sigma_g)$, choose $\tilde{\phi} \in \mathcal{M}_{g,1}$ and $\gamma \in \pi_1(\Sigma_g^0)$ such that $\tilde{\phi}$ and γ project to ϕ and x respectively. Then we set $f_0(\phi, x)=d(\tilde{\phi}(\gamma))-d(\gamma)$. Now define a map $k_0: \mathcal{M}_{g,*}\to H_1(\Sigma_g)$ by the following property

$$k_0(\phi)\cdot x=f_0(\phi, x) \qquad \text{for all } x \in H_1(\Sigma_g).$$

It is easy to see that $k_0(\phi\psi)=\psi_*^{-1}(k_0(\phi))+k_0(\psi)$ for all $\phi, \psi \in \mathcal{M}_{g,*}$. We should mention here that Earle [1] has already constructed a crossed homomorphism $\mathcal{M}_{g,*}\to H_1(\Sigma_g; (1/(2g-2))\boldsymbol{Z})$ by a complex analytic method. It can be shown that our crossed homomorphism

$$\frac{1}{2-2g}k_0: \mathcal{M}_{g,*}\longrightarrow H_1\left(\Sigma_g;\ \frac{1}{2g-2}\boldsymbol{Z}\right)$$

coincides with his one up to a coboundary.

Now we can define a homomorphism

$$\rho'(f_0): \mathcal{M}_{g,*}\longrightarrow H_1\left(\Sigma_g;\ \frac{1}{2g-2}\boldsymbol{Z}\right)\rtimes Sp(2g;\boldsymbol{Z})$$

by $\rho'(f_0)\ (\phi)=((1/(2-2g))\phi_*(k_0(\phi)), \rho(\phi))$, where $\rho: \mathcal{M}_{g,*}\to Sp(2g;\boldsymbol{Z})$ is the natural homomorphism. It turns out that if ϕ is contained in $\pi_1(\Sigma_g)\subset\mathcal{M}_{g,*}$, then $k_0(\phi)=(2-2g)\ [\phi]$. Hence $\rho'(f_0)$ induces a homomorphism

$$\overline{\rho'(f_0)}: \mathcal{M}_g\longrightarrow H_1(\Sigma_g;\boldsymbol{Z}/2g-2)\rtimes Sp(2g;\boldsymbol{Z})$$

and we obtain

Theorem 1. *We have the following commutative diagram*

$$\begin{array}{ccc}
\mathcal{M}_{g,*} & \xrightarrow{\rho'(f_0)} & H_1(\Sigma_g;(1/(2g-2))\boldsymbol{Z})\rtimes Sp(2g;\boldsymbol{Z}) \\
\downarrow & & \downarrow H_1(\Sigma_g;\boldsymbol{Z}) \\
\mathcal{M}_g & \xrightarrow[\overline{\rho'(f_0)}]{} & H_1(\Sigma_g;\boldsymbol{Z}/2g-2)\rtimes Sp(2g;\boldsymbol{Z}).
\end{array}$$

Corollary 2 (Topological version of Earle's embedding theorem). *For any oriented Σ_g-bundle $\pi: E\to X$, there exists a flat T^{2g}-bundle $\pi': J'\to X$ with structure group $(\boldsymbol{Z}/2g-2)^{2g}\rtimes Sp(2g;\boldsymbol{Z})\subset T^{2g}\rtimes Sp(2g;\boldsymbol{Z})$ and a fibre preserving map $j': E\to J'$ such that the restriction of j' to each fibre induces an isomorphism on the first integral homology.*

3. New relations among characteristic classes of surface bundles. As in [3] we write $\overline{Sp}(2g;\boldsymbol{Z})$ for the semi-direct product $H_1(\Sigma_g)\rtimes Sp(2g;\boldsymbol{Z})$. Let $\rho(f_0): \mathcal{M}_{g,*}\to\overline{Sp}(2g;\boldsymbol{Z})$ be the homomorphism defined by $\rho(f_0)\ (\phi)=(\phi_*(k_0(\phi)), \rho(\phi))$ and let $\Omega \in H^2(\overline{Sp}(2g;\boldsymbol{Z});\boldsymbol{Z})$ be the cohomology class represented by the 2-cocycle $((x, A), (y, B))\to x\cdot Ay\ (x, y \in H_1(\Sigma_g), A, B \in Sp(2g;\boldsymbol{Z}))$. Then we have

Theorem 3. $\rho(f_0)^*(\Omega)=2g(2-2g)e-e_1 \in H^2(\mathcal{M}_{g,*};\boldsymbol{Z})$.

Remark 4. It would be an interesting problem to determine whether an appropriate holomorphic version of the above theorem holds or not (compare Remark 8-1 of [3]). One way to attack this problem would be to examine certain holomorphic line bundles over the universal Teichmüller curve.

Now let $\overline{\mathcal{M}}_{g,*}$ be the semi-direct product $\pi_1(\Sigma_g)\rtimes\mathcal{M}_{g,*}$ and let $\bar{\pi}:\overline{\mathcal{M}}_{g,*}\to\mathcal{M}_{g,*}$ be the homomorphism defined by $\bar{\pi}((\gamma,\phi))=\gamma\phi$ (see [3]). The composition $\rho(f_0)\bar{\pi}:\overline{\mathcal{M}}_{g,*}\to\overline{Sp}(2g;\boldsymbol{Z})$ is *not* the same as the natural homomorphism $\bar{\rho}:\overline{\mathcal{M}}_{g,*}\to\overline{Sp}(2g;\boldsymbol{Z})$ given by $\bar{\rho}((\gamma,\phi))=([\gamma],\rho(\phi))$. This means the following. Namely suppose that there is given a surface bundle $\pi:E\to X$ with a cross section $s:X\to E$. Then we have two flat T^{2g}-bundles $\pi:J\to X$ defined in [3] and $\pi':J'\to X$ defined in this note. Although they are isomorphic as differentiable T^{2g}-bundles, they are not isomorphic as *flat* T^{2g}-bundles in general. We can go even further. First observe that to any crossed homomorphism $f:\overline{\mathcal{M}}_{g,*}\times H_1(\Sigma_g)\to\boldsymbol{Z}$, there is associated a homomorphism $\rho(f):\overline{\mathcal{M}}_{g,*}\to\overline{Sp}(2g;\boldsymbol{Z})$ and the cohomology class $\rho(f)^*(\Omega)\in H^2(\overline{\mathcal{M}}_{g,*};\boldsymbol{Z})$ depends only on the cohomology class of f. Now it can be shown that $H^1(\overline{\mathcal{M}}_{g,*};H^1(\Sigma_g))$, namely the set of all cohomology classes of crossed homomorphisms $\overline{\mathcal{M}}_{g,*}\times H_1(\Sigma_g)\to\boldsymbol{Z}$, is isomorphic to $\boldsymbol{Z}^2$ whose generators can be given as follows. Let $f_1:\overline{\mathcal{M}}_{g,*}\times H_1(\Sigma_g)\to\boldsymbol{Z}$ be the crossed homomorphism defined by $f_1((\gamma,\phi),x)=(\phi_*^{-1}([\gamma]))\cdot x$ and let $\pi:\overline{\mathcal{M}}_{g,*}\to\mathcal{M}_{g,*}$ be the homomorphism given by $\pi((\gamma,\phi))=\phi$. Then the cohomology classes $[\pi^*(f_0)]$ and $[f_1]$ form a basis of $H^1(\overline{\mathcal{M}}_{g,*};H^1(\Sigma_g))$.

Theorem 5. *For each crossed homomorphism* $f=m\pi^*(f_0)+nf_1:\overline{\mathcal{M}}_{g,*}\times H_1(\Sigma_g)\to\boldsymbol{Z}$ $(m,n\in\boldsymbol{Z})$, *let* $\rho(f):\overline{\mathcal{M}}_{g,*}\to\overline{Sp}(2g;\boldsymbol{Z})$ *be the associated homomorphism. Then we have*

$$\begin{aligned}\rho(f)^*(\Omega)=&(2n^2-4mn+4mng)\nu+\{m^2 2g(2-2g)-n^2+2mn-4mng\}\pi^*(e)\\&-(n^2-2mn)\bar{\pi}^*(e)-m^2e_1 \quad \text{in } H^2(\overline{\mathcal{M}}_{g,*};\boldsymbol{Z}).\end{aligned}$$

If we apply the argument of [2], [3] to the above, we obtain

Corollary 6. (i) *Any polynomial of* ν, $\pi^*(e)$, $\bar{\pi}^*(e)$ *and* e_1 *of degree* $2(g+1)$ *vanishes in* $H^{2(g+1)}(\overline{\mathcal{M}}_{g,*};\boldsymbol{Q})$.

(ii) $e_1^i e_j e^k$ *vanishes in* $H^{2g}(\mathcal{M}_{g,*};\boldsymbol{Q})$ *for any* $i,j,k\geqq 0$ *with* $i+j+k=g$.

(iii) $e_1^i e_j e_k$ *vanishes in* $H^{2(g-1)}(\mathcal{M}_g;\boldsymbol{Q})$ *for any* $i,j,k\geqq 0$ *with* $i+j+k=g-1$.

It takes a good deal of lengthy calculations to figure out the quotient of our characteristic classes divided by all the relations obtained so far. However in view of a few computations for small g, we might say that we have already obtained relatively large part of the whole relations.

The details of the results sketched in this note will appear elsewhere.

References

[1] C. J. Earle: Families of Riemann surfaces and Jacobi varieties. Ann. of Math., **107**, 255–286 (1978).
[2] S. Morita: Family of Jacobian manifolds and characteristic classes of surface bundles. Proc. Japan Acad., **60**A, 373–376 (1984).
[3] ——: Family of Jacobian manifolds and characteristic classes of surface bundles (preprint).