1. A Study of a Certain Non-Conventional Operator of Principal Type. II

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Introduction. We continue our study of the operator

(1) $$B^I = D_t + \sqrt{-1}(t^2/2 + x)D_y,$$

$D_t = -\sqrt{-1}\partial/\partial t$, $D_y = -\sqrt{-1}\partial/\partial y$, in (a neighborhood of the origin in) $\boldsymbol{R}^3$ ([3]). Here we discuss solvability of the equation:

(2) $$B^I u = f$$

for a given f. Since the operator B^I is not locally solvable, our primary task is to specify the conditions on f which guarantee existence of a solution u to (2). One such condition is Condition $(A^{\pm})$ to be introduced in the next section (see also Theorem in § 2).

1. Condition $A^{\pm}$. Let $\beta(t, r, x) = \int_r^t (s^2/2 + x)ds$. Denote by $\tilde{f}$ the Fourier transform of f with respect to the argument y provided it makes sense. Define

(3) $$J^{\pm}(f\,; x, \eta) = \int_{\pm\infty}^{\mp\sqrt{-2x}} \tilde{f}(r, x, \eta) \exp\{\pm\beta(\pm\sqrt{-2x}, r, x)\eta\} dr$$

if $x<0$ and $\pm\eta>0$. Note $\beta(\pm\sqrt{-2x}, r, x)\eta \leqq 0$ in the integrals. We set $J^{\pm}(f\,; x, \eta) = 0$ for $x \geqq 0$ or for $x<0$ and $\pm\eta<0$. We write $J_k^{\pm}(x, \eta)$ instead of $J^{\pm}(f_k\,; x, \eta)$, where $f_k^{\pm} = (t \mp \sqrt{-2x})^k$.

Lemma 1. *For any $x<0$, $\pm\eta>0$ and $m, n = 0, 1, 2, \cdots$, we have*

$$|\partial_\eta^m (x\partial_x)^n J_k^{\pm}(x, \eta)| \leqq C|\eta|^{-(k+1)/3-m}(1+|\eta|(\sqrt{-2x})^3)^{(m+n)/3};$$

$J_0^{\pm}(x, \eta) > 0$ and

$$|\partial_\eta^m (x\partial_x)^n \{J_0^{\pm}(x, \eta)^{-1}\}| \leqq C|\eta|^{1/3-m}(1+|\eta|(\sqrt{-2x})^3)^{1/6+2(m+n)/3}.$$

Here C stands for various constants.

This lemma can be proved by a routine computation. $J_k^{\pm}(x, \eta)$ can be expressed in terms of confluent hypergeometric functions and related functions. For details, see [4].

Now we have to choose the class of functions $f(t, x, y)$ for which the integrals (3) are well-defined. Let $\mathcal{F}$ be the class of distributions $f(t, x, y)$ in $\mathcal{S}'(\boldsymbol{R}^3)$ such that for each $h(y)$ in $\mathcal{S}(\boldsymbol{R}_y)$ the coupling $\langle f(t, x, y), h(y)\rangle$ is continuous in t, at most of polynomial growth in t, and measurable in x. Decompose $f \in \mathcal{F}$ into a difference: $f = f^+ - f^-$, where $\tilde{f}^{\pm}$ are supported in $\pm\eta>0$ so that $f^{\pm}$ have holomorphic extensions in $\pm \operatorname{Im} y > 0$. Denote by $\mathcal{F}^{\pm}$ the sets of $f^{\pm}$. Then $\mathcal{F}^{\pm}$ are subspaces of $\mathcal{F}$ and $\mathcal{F} = \mathcal{F}^+ - \mathcal{F}^-$ holds in an obvious manner.

Lemma 2. *Let*

(4) $$(Q^{\pm}f)^{\sim}(x, \eta) = J^{\pm}(f\,; x, \eta)/J_0^{\pm}(x, \eta)$$

if $x<0$ *and* $\pm\eta>0$ *while* $(Q^{\pm}f)^{\sim}(x,\eta)=0$ *if* $x\geqq 0$ *or if* $x<0$ *and* $\pm\eta<0$. *For any* $f=f^{+}-f^{-}\in\mathcal{F}$, *we have decompositions*:

$$f^{\pm}(t,x,y)=f_0^{\pm}(t,x,y)+f_1^{\pm}(x,y)$$

such that $Q^{\pm}f_0^{\pm}=0$, $Q^{\pm}f_1^{\pm}=f_1^{\pm}$, $Q^{\pm}f_1^{\mp}=0$.

In fact, $f_0^{\pm}=f^{\pm}-Q^{\pm}f$, $f_1^{\pm}=Q^{\pm}f$. Note $Q^{\pm}f^{\mp}=0$.

Definition. We say that $f\in\mathcal{F}$ satisfies Condition $(A^{\pm})$ if

$$|\langle\psi(x,\eta),\ J^{\pm}(f\,;x,\eta)\exp\{\mp 4x\sqrt{-2x}\eta/3\}\rangle| \leqq C\sup_{x,\eta}\textstyle\sum_{j+k+m\leqq N}|(x\partial_x)^j\eta^k\partial_\eta^m\psi(x,\eta)|$$

for $\psi\in\mathcal{S}(\boldsymbol{R}^2_{x,\eta})$. Here N is a suitable positive integer, and C is a positive constant.

Note $\pm 4x\sqrt{-2x}/3=\beta(\pm\sqrt{-2x},\mp\sqrt{-2x},x)$, $x<0$, and $\pm\beta(\mp\sqrt{-2x},r,x)>0$ for $-\sqrt{-2x}<\pm r<2\sqrt{-2x}$, $x<0$. Thus, Condition $(A^{\pm})$ roughly provides a control of the behaviors of $\tilde{f}(t,x,\eta)$ with respect to η for $-\sqrt{-2x}<\pm t<2\sqrt{-2x}$, $x<0$. However, the condition itself is too involved. We indicate some of its flavors.

Proposition. *Let* $f(t,x,y)$ *be such that* $\tilde{f}(t,x,\eta)$ *is smooth and satisfies*

$$|\partial_t^m\tilde{f}(t,x,\eta)|\leqq C(1+|\eta|)^s,\qquad m=0,1,$$

for some s. *If, for large* η, $\tilde{f}(t,x,\eta)$ *is positively homogeneous in* η, *and if* f *satisfies Condition* $(A^{\pm})$, *then* $f(\pm\sqrt{-2x},x,y)$, $x<0$, *have holomorphic extensions with respect to* y *in* $\pm\operatorname{Im}y<0$.

In fact, $\tilde{f}^{\pm}(t,x,\eta)=\tilde{f}^{\pm}(\pm\sqrt{-2x},x,\eta)+(t\mp\sqrt{-2x})\tilde{g}^{\pm}(t,x,\eta)$ by Taylor's expansion, Lemma 1 implies

$$(Q^{\pm}f)^{\sim}(x,\eta)=\tilde{f}^{\pm}(\pm\sqrt{-2x},x,\eta)+O(|\eta|^{s-1/6}),\qquad \pm\eta>0,$$

$x<0$, and $\tilde{f}^{\pm}(\pm\sqrt{-2x},x,\eta)\exp\{\mp 4x\sqrt{-2x}\eta/3\}$ tempered in η, $\pm\eta>0$. It follows $\tilde{f}^{\pm}(\pm\sqrt{-2x},x,\eta)=0$, $x<0$. Since $f=f^{+}-f^{-}$, $f(\pm\sqrt{-2x},x,y)=f^{\mp}(\pm\sqrt{-2x},x,y)$ are holomorphic in $\pm\operatorname{Im}y<0$.

To characterize the range of the operator B^I, F. Treves has speculated a condition of holomorphic extendability of the restrictions of f to $t^2/2+x=0$, $x<0$, in connection with a general framework N. Hanges and F. Treves have been developing ([1], [2]). The above proposition confirms albeit to a limited extent a part of Treves' speculation. Actually we expect that the decompositions:

$$(5)\qquad J^{\pm}(f\,;x,\eta)=(A^{\pm}f)^{\sim}(x,\eta)+(B^{\pm}f)^{\sim}(x,\eta)\exp\{\pm 4x\sqrt{-2x}\eta/3\}$$

be valid for a wide class of functions $f(t,x,y)$, where $(A^{\pm}f)(x,y)$ and $(B^{\pm}f)(x,y)$ are tempered with respect to y. This is in fact true if $f(t,x,y)$ is a polynomial in t. However, we do not know the exact extent of validity of (5). For those f satisfying (5) Condition $(A^{\pm})$ means that $(A^{\pm}f)(x,y)$ have holomorphic extensions with respect to y in $4x\sqrt{-2x}/3<\pm\operatorname{Im}y$, $x<0$.

2. Main results. Now we state and prove the following

Theorem. *Assume* $f(t,x,y)$ *satisfy Condition* $(A^{\pm})$. *Then the equation* (2) *has a solution* $u(t,x,y)$ *such that* u *and* u_t *both belong to the class* $\mathcal{F}$. *Conversely, if* u *and* $u_t\in\mathcal{F}$, *then* B^Iu *satisfies Condition* $(A^{\pm})$.

Here is a very easy proof. Fourier transforming (2) with respect to

y, we get

(6) $$\{D_t+\sqrt{-1}(t^2/2+x)\eta\}\tilde{u}=\tilde{f},$$

or

(7) $$D_t(\tilde{u}\exp\{-\beta(t,s,x)\eta\})=\tilde{f}\exp\{-\beta(t,s,x)\eta\}$$

for any s and η. First we show the second half of Theorem. If u and $u_t \in \mathcal{F}$, then substituting $f=B^I u$ into (3) with $s=\pm\sqrt{-2x}$, we get

(8) $$J^{\pm}(B^I u\,;x,\eta)=\sqrt{-1}\tilde{u}(\mp\sqrt{-2x},x,\eta)\exp\{\beta(\pm\sqrt{-2x},\mp\sqrt{-2x},x)\eta\}$$

when $x<0$, $\pm\eta>0$. Therefore, $B^I u$ satisfies Condition $(A^{\pm})$.

Now we show the first half of Theorem. Let f satisfy Condition $(A^{\pm})$. Recall the decomposition $f=f^+-f^-$. We decompose u analogously: $u=u^+-u^-$. Then (6) and (7) are valid with $\tilde{u}$ and $\tilde{f}$ replaced by $\tilde{u}^{\pm}$ and $\tilde{f}^{\pm}$. (6), (7) thus replaced are still called (6), (7). Restricting the domains of integrations to where $\beta\eta\leqq 0$, we get from (7)

(9) $$\tilde{u}^{\pm}(t,x,\eta)=\sqrt{-1}\int_{\pm\infty}^{t}\tilde{f}^{\pm}(r,x,\eta)\exp\{\beta(t,r,x)\eta\}dr$$

when $x\geqq 0$ or $x<0$ and $\pm t\geqq\sqrt{-2x}$, and

(10) $$\tilde{u}^{\pm}(t,x,\eta)=\sqrt{-1}\int_{\mp\sqrt{-2x}}^{t}\tilde{f}^{\pm}(r,x,\eta)\exp\{\beta(t,r,x)\eta\}dr$$
$$+\tilde{u}^{\pm}(\mp\sqrt{-2x},x,\eta)\exp\{\beta(t,\mp\sqrt{-2x},x)\eta\}$$

when $x<0$ and $\pm t<\sqrt{-2x}$. Substituting (9) and (10) into (6), we obtain the jump condition at $t=\pm\sqrt{-2x}$, $x<0$:

(11) $$J^{\pm}(f^{\pm}\,;x,\eta)=\sqrt{-1}\tilde{u}^{\pm}(\mp\sqrt{-2x},x,\eta)\exp\{\beta(\pm\sqrt{-2x},\mp\sqrt{-2x},x)\eta\}.$$

(10) is thus rewritten:

(12) $$\tilde{u}^{\pm}(t,x,\eta)=\sqrt{-1}\int_{\mp\sqrt{-2x}}^{t}\tilde{f}^{\pm}(r,x,\eta)\exp\{\beta(t,r,x)\eta\}dr$$
$$+\sqrt{-1}J^{\pm}(f^{\pm}\,;x,\eta)\exp\{\beta(t,\pm\sqrt{-2x},x)\eta\}.$$

Therefore, if Condition $(A^{\pm})$ holds, then (9) and (10) lead to a solution of (2).

References

[1] N. Hanges and F. Treves: (in preparation).
[2] F. Treves: (Personal communication).
[3] A. Yoshikawa: A study of a certain non-conventional operator of principal type. Proc. Japan Acad., 60A, 90–92 (1984).
[4] ——: On the evaluation of certain phase integrals (in preparation).