19. Representations over G-Rings and Cohomology*)

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(Communicated by Kunihiko KODAIRA, M. J. A., Feb. 13, 1984)

§1. **Introduction.** Let G be a group. The word *ring* will always mean associative ring with an identity element 1. A *G-ring* is a ring Λ together with a G-action on Λ preserving the ring structure. Then we introduce a Grothendieck group $R(G, \Lambda)$ associated with the abelian semi-group consisting of representations over Λ. The group $R(G, \Lambda)$ is a generalization of the representation rings $R(G)$ and $RO(G)$.

The purpose of the present paper is to express $R(G, \Lambda)$ in terms of the cohomology $H^1(G; GL(n, \Lambda))$ of the group G with coefficients in a non-abelian group $GL(n, \Lambda)$ in the sense of Serre [3].

In some cases, $R(G, \Lambda)$ is isomorphic to an equivariant algebraic K-group $K^G(\Lambda; F_f)_d$ and we can express $K^G(\Lambda; F_f)_d$ in terms of the cohomology $H^1(G; GL(n, \Lambda))$. An interesting example is provided by Serre [3]. In fact the example was a starting point of the present investigation.

The consideration of the present paper will be used to prove an induction theorem for equivariant K-theory in a subsequent paper [2].

§2. **$R(G, \Lambda)$.** Let Λ be a G-ring. A *ΛG-module* is a module M over Λ together with a G-action on M such that

(*) $$g(\lambda_1 m_1 + \lambda_2 m_2) = (g\lambda_1)(gm_1) + (g\lambda_2)(gm_2)$$

for any $g \in G$, $\lambda_i \in \Lambda$, $m_i \in M$. In this paper any modules are assumed to be finitely generated.

Then $R(G, \Lambda)$ is defined to be the abelian group given by generators $[M]$ where M is a ΛG-module which is free over Λ, with relations

$$[M] = [M'] + [M'']$$

whenever $M \cong M' \oplus M''$.

When Λ is a commutative G-ring, we can consider a product $M_1 \otimes M_2$ of two ΛG-modules M_1, M_2 (see [1]). If M_1, M_2 are free over Λ, $M_1 \otimes M_2$ is also free over Λ. Hence this product induces a structure of commutative ring in $R(G, \Lambda)$.

Remark 2.1. If Λ is $\boldsymbol{R}$ (the field of the real numbers) or $\boldsymbol{C}$ (the field of the complex numbers) with trivial G-action, then $R(G, \Lambda)$ is nothing but the real representation ring $RO(G)$ or the complex representation ring $R(G)$ respectively.

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*) Dedicated to Prof. Minoru Nakaoka on his sixtieth birthday. Research supported in part by Grant-in-Aid for Scientific Research.

§3. **$H^1(G; GL(n, \Lambda))$.** Let us recall the first cohomology $H^1(G; \Gamma)$ of Serre [3]. A *G-group* is a group Γ together with a G-action on Γ preserving the group structure. Then a map $A: G \to \Gamma$ is called a *cocycle* if the following equality holds:

$$A(gg') = A(g) \cdot (g \cdot A(g')) \qquad \text{for any } g, g' \in G.$$

Set

$$Z^1(G; \Gamma) = \{A: G \to \Gamma \text{ cocycle}\}.$$

Two elements A and B of $Z^1(G; \Gamma)$ are *cohomologous* (denoted by $A \sim B$) if and only if there exists an element $C \in \Gamma$ such that

$$B(g) = C^{-1} \cdot A(g) \cdot (g \cdot C) \qquad \text{for any } g \in G.$$

Then the relation $\sim$ is an equivalence relation and the first cohomology $H^1(G; \Gamma)$ of G with coefficients in Γ is defined to be the quotient set

$$H^1(G; \Gamma) = Z^1(G; \Gamma)/\sim.$$

The equivalence class including A is denoted by $[A]$.

Notice that if Γ is non-abelian, there is no canonical group structure on $H^1(G; \Gamma)$ inherited from G and Γ in general. There is however a *distinguished* element represented by $A_0: G \to \Gamma$ with $A_0(g) = e$ (unit element of Γ) for all $g \in G$.

Let $GL(n, \Lambda)$ be the group of invertible $n \times n$ matrices over a G-ring Λ. The G-action on each entry of a matrix induces a G-action on $GL(n, \Lambda)$, which makes $GL(n, \Lambda)$ a G-group.

Theorem 3.1. *Let M be a free Λ-module of rank n. Then the isomorphism classes of ΛG-module structures on M are in one to one correspondence with $H^1(G; GL(n, \Lambda))$.*

Outline of proof. Taking an arbitrary base of M, a G-action on M is expressed by a map

$$A: G \longrightarrow GL(n, \Lambda).$$

One verifies that G acts on M satisfying the condition $(*)$ in §2 if and only if A is a cocycle. Furthermore two cocycles A and B represent isomorphic ΛG-modules if and only if A and B are cohomologous.

§4. **Grothendieck group.** Denote by

$$(**) \qquad \coprod_{n \geq 0} H^1(G; GL(n, \Lambda))$$

the disjoint union of cohomologies $H^1(G; GL(n, \Lambda))$ where we set $H^1(G; GL(0, \Lambda)) = \{0\}$. An abelian semi-group structure is imposed on it as follows. Let $A: G \to GL(m, \Lambda)$ (resp. $B: G \to GL(n, \Lambda)$) represent an arbitrary element of $H^1(G; GL(m, \Lambda))$ (resp. $H^1(G; GL(n, \Lambda))$). Then we define $D: G \to GL(m+n, \Lambda)$ by

$$D(g) = \begin{pmatrix} A(g) & 0 \\ 0 & B(g) \end{pmatrix}.$$

Clearly D is a cocycle and the assignment $(A, B) \mapsto D$ induces a map

$$\Phi: H^1(G; GL(m, \Lambda)) \times H^1(G; GL(n, \Lambda)) \longrightarrow H^1(G; GL(m+n, \Lambda)).$$

One verifies that $\Phi([A], [B]) = \Phi([B], [A])$ and that Φ gives an abelian semi-group structure in the set (**) above. The Grothendieck group associated with the abelian semi-group above is denoted by

$$(***)\qquad K(\coprod_{n\geq 0} H^1(G\,;GL(n,\Lambda))).$$

When Λ is a commutative G-ring, we define $D': G\to GL(mn,\Lambda)$ by

$$D'(g)=A(g)\otimes B(g)$$

the tensor product of the matrices $A(g)$ and $B(g)$ for $g\in G$. Then one verifies that D' is a cocycle. Moreover it is easy to see that the assignment $(A,B)\mapsto D'$ induces a map

$$\Psi: H^1(G\,;GL(m,\Lambda))\times H^1(G\,;GL(n,\Lambda))\longrightarrow H^1(G\,;GL(mn,\Lambda))$$

and that Φ and Ψ give a commutative semi-ring structure in the set (**) above. Hence the Grothendieck group (***) has an induced commutative ring structure.

If the ring Λ is such that, given $m,n>0$, $\Lambda^m\cong\Lambda^n$ (forgetting G-action) only if $m=n$, we say that Λ has *invariant basis number* (abbreviated IBN).

Theorem 4.1. *If Λ has IBN, then we have an isomorphism*

$$R(G,\Lambda)\cong K(\coprod_{n\geq 0} H^1(G\,;GL(n,\Lambda)))$$

of abelian groups. When Λ is commutative, both terms have commutative ring structures and $\cong$ stands for a ring isomorphism.

Proof. Theorem 4.1 follows easily from Theorem 3.1.

§5. Equivariant algebraic K-theory and examples.

In [1], we introduced two kinds of equivariant algebraic K-groups $K^G(\Lambda\,;F)_d$ and $K^G(\Lambda\,;F)_e$ for each family F of ΛG-modules.

We now give examples of families:

$F_a=\{$all ΛG-modules$\}$

$F_f=\{$all ΛG-modules which are free over $\Lambda\}$

$F_{tf}=\{$all torsion free ΛG-modules$\}$.

Lemma 5.1. *If a G-ring Λ is such that every projective module over Λ is stably free, then we have an isomorphism*

$$K^G(\Lambda\,;F_f)_d\cong R(G,\Lambda)$$

of abelian groups. When Λ is commutative, both terms have commutative ring structures and $\cong$ stands for a ring isomorphism.

By combining Theorem 4.1 and Lemma 5.1, we have

Theorem 5.2. *Under the condition of Lemma* 5.1, *we have an isomorphism*

$$K^G(\Lambda\,;F_f)_d\cong K(\coprod_{n\geq 0} H^1(G\,;GL(n,\Lambda)))$$

of abelian groups. When Λ is commutative, both terms have commutative ring structures and $\cong$ stands for a ring isomorphism.

As examples satisfying the condition of Lemma 5.1, we have

Proposition 5.3. *If a G-ring Λ is a field, a skew field, a principal ideal domain, or a local ring, we have*

$$K^G(\Lambda; F_f)_d \cong R(G, \Lambda) \cong K(\coprod_{n\geq 0} H^1(G; GL(n, \Lambda))).$$

Let K/k be a Galois extension and G be the Galois group of K/k. Then K is a G-ring in our sense and we have

Corollary 5.4. $K^G(K; F_a)_d \underset{(I)}{\cong} K^G(K; F_{tf})_d \underset{(II)}{\cong} K^G(K; F_f)_d \underset{(III)}{\cong} R(G, K) \underset{(IV)}{\cong} Z$. *Here Z denotes the group of integers. If the characteristic* (*char K*) *of K is zero or* (*char K*, $|G|)=1$, *then d in the formula can be replaced by e. Here $|G|$ denotes the order of G.*

Proof. According to Serre [3], the first cohomology $H^1(G; GL(n, K))$ vanishes for all $n \geqq 1$. It follows that the abelian semi-group (**) in § 4 is isomorphic to the semi-group of non-negative integers. Hence the Grothendieck group (***) in § 4 is isomorphic to Z. Accordingly the isomorphisms (III) and (IV) follow from Proposition 5.3, while the isomorphisms (I) and (II) are easy to prove. If char K is zero or (char K, $|G|)=1$, then every short exact sequence of KG-modules is split exact. Hence the relations to define $K^G(K; F)_d$ and $K^G(K; F)_e$ are equivalent in this case.

References

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