17. On the Prolongation of Solutions for Quasilinear Differential Equations

By Takao KOBAYASHI
Department of Mathematics, Tokyo Metropolitan University*)

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§0. Introduction. It is well known (M. Zerner [4]) that holomorphic solutions of linear differential equations are holomorphically continued across non-characteristic surfaces. Y. Tsuno [3] showed that this is true for quasilinear equations if the derivatives of solutions up to order $m+1$ are bounded, where m is the order of equation. T. Ishii [1] has recently constructed solutions (for semilinear equations) which are singular along non-characteristic surfaces (see also [2]). So some boundedness conditions are necessary, in general. There is, however, a gap in boundedness properties of solutions between their results.

The aim of this note is to bridge the gap. To do this we show that for each equation there is an exponent σ less than or equal to $m-1$ and that boundedness of order up to σ is sufficient for prolongation.

§1. Definitions. Let Ω be a domain in C^n containing the origin and Λ be the set of multi-indices $\{\beta\in(Z_+)^n : |\beta|\leqq m-1\}$. The variables in C^n and C^N with $N=\sharp\Lambda$ are denoted by $z=(z_1,\cdots,z_n)$ and $p=(p_\beta)_{\beta\in\Lambda}$, respectively. We consider the following quasilinear differential equation:

(1) $$\sum_{|\alpha|=m} a_\alpha(z,(D^\beta u))D^\alpha u=b(z,(D^\beta u)),$$

where $(D^\beta u)=(D^\beta u)_{\beta\in\Lambda}$ with $D=\partial/\partial z$ and $a_\alpha(z,p)$, $b(z,p)$ are holomorphic functions on $\Omega\times C^N$.

Let ϕ be a real-valued C^1 function on Ω with $\phi(0)=0$. We put $\Omega_-=\{z\in\Omega:\phi(z)<0\}$ and $\partial\Omega_-=\{z\in\Omega:\phi(z)=0\}$.

We ask whether u is holomorphic in a neighborhood of the origin if u is holomorphic in Ω_- and satisfies (1).

We assume that $\partial\Omega_-$ is non-characteristic at the origin, that is,

(A) $$\zeta^0=\mathrm{grad}_z\,\phi(0)\neq 0,\qquad \sum_{|\alpha|=m} a_\alpha(z,p)(\zeta^0)^\alpha\neq 0 \quad \text{for } (z,p)\in\Omega\times C^N.$$

Under the condition (A) we may assume that $\zeta^0=(1,0,\cdots,0)$ and can rewrite (1) as

(2) $$D_1^m u=\sum_{\substack{|\alpha|=m\\ \alpha\neq(m,0,\cdots,0)}} a_\alpha(z,(D^\beta u))D^\alpha u+b(z,(D^\beta u)),$$

*) Fukasawa, Setagaya-ku, Tokyo, 158 Japan.

where $a_\alpha(z,p)$, $b(z,p)$ are holomorphic on $\Omega\times C^N$ (different from those in (1)). We expand $a_\alpha(z,p)$, $b(z,p)$ with respect to p:

$$(3)\qquad a_\alpha(z,p)=\sum_\mu a_{\alpha,\mu}(z)p^\mu, \qquad b(z,p)=\sum_\mu b_\mu(z)p^\mu,$$

where $\mu=(\mu_\beta)_{\beta\in\Lambda}\in(Z_+)^N$ is a multi-index and $a_{\alpha,\mu}(z)$, $b_\mu(z)$ are holomorphic in Ω. We put

$$(4)\qquad \begin{aligned}\{a\}&=\{\nu\in(Z_+)^N : |\nu|\geqq 1,\ {}^\exists\mu\geqq\nu,\ {}^\exists\alpha \text{ s.t. } a_{\alpha,\mu}(z)\neq 0\},\\ \{b\}&=\{\nu\in(Z_+)^N : |\nu|\geqq 2,\ {}^\exists\mu\geqq\nu \text{ s.t. } b_\mu(z)\neq 0\},\end{aligned}$$

where $|\mu|=\sum_{\beta\in\Lambda}\mu_\beta$. If $\{a\}$ (resp. and $\{b\}$) is empty, (2) is semilinear (resp. linear).

Definition. For the equation (2) we define exponents σ_a and σ_b as follows:

$$(5)\qquad \begin{aligned}\sigma_a&=\sup_{\nu\in\{a\}}\frac{\langle\nu,|\Lambda|\rangle}{|\nu|}, && \text{if } \{a\}\neq\phi,\\ \sigma_b&=\sup_{\nu\in\{b\}}\frac{\langle\nu,|\Lambda|\rangle-m}{|\nu|-1}, && \text{if } \{b\}\neq\phi,\end{aligned}$$

and if $\{a\}$ (resp. $\{b\}$) is empty, we put $\sigma_a=-\infty$ (resp. $\sigma_b=-\infty$), where $\langle\nu,|\Lambda|\rangle=\sum_{\beta\in\Lambda}\nu_\beta\times|\beta|$.

Note that since

$$0\leqq\frac{\langle\nu,|\Lambda|\rangle}{|\nu|}\leqq m-1 \quad\text{and}\quad -m\leqq\frac{\langle\nu,|\Lambda|\rangle-m}{|\nu|-1}<m-1$$

we have $\sigma_a=-\infty$ or $0\leqq\sigma_a\leqq m-1$ and $\sigma_b=-\infty$ or $-m\leqq\sigma_b\leqq m-1$.

Definition. For $\sigma\leqq m-1$ we say that $u\in\mathcal{O}(\Omega_-)$ is bounded of order σ (resp. σ_{+0}) in Ω_- if u satisfies (6) and (7) (resp. (6) and (8)):

$$(6)\qquad \sup_{z\in\Omega_-}|D^\beta u(z)|\leqq M,\ |\beta|\leqq\sigma, \quad\text{with}\quad M>0.$$

$$(7)\qquad \sup_{\substack{z'\\(-\varepsilon,z')\in\Omega_-}}|D^\beta u(-\varepsilon,z')|\leqq O(\varepsilon^{\sigma-|\beta|}),\ \sigma<|\beta|\leqq m-1, \quad\text{as } \varepsilon \text{ tends to } 0.$$

$$(8)\qquad \sup_{\substack{z'\\(-\varepsilon,z')\in\Omega_-}}|D^\beta u(-\varepsilon,z')|\leqq o(\varepsilon^{\sigma-|\beta|}),\ \sigma<|\beta|\leqq m-1, \quad\text{as } \varepsilon \text{ tends to } 0.$$

§2. Results.

Our main result is

Theorem. *We assume* (A) *to hold. Let σ_a and σ_b be the exponents defined by* (5). *Let u be holomorphic in Ω_- and satisfy* (2). *In addition we assume that one of* (B.1)–(B.4) *holds*:

(B.1) *u is bounded of order $m-1$ if* $\max(\sigma_a,\sigma_b)=m-1$.

(B.2) *u is bounded of order σ for some σ, $\sigma_b<\sigma<m-1$ if $\sigma_a\leqq\sigma_b<m-1$.*

(B.3) *u is bounded of order σ_{a+0} if $\sigma_b<\sigma_a$.*

(B.4) *u is arbitrary if $\sigma_a=\sigma_b=-\infty$.*

Then u is holomorphic in a neighborhood of the origin.

Remark. In any case the assertion is true if u is bounded of order $m-1$.

Remark. T. Ishii [1] has constructed solutions which are singular along non-characteristic surfaces and bounded of order σ_b for semilinear equations whose nonlinear term is polynomial in p.

Example. The exponents of the equation

$$\partial^2 u/\partial z_1^2 = \exp(-\partial u/\partial z_1) \quad \text{in } \boldsymbol{C}^n \tag{9}$$

are $\sigma_a = -\infty$ and $\sigma_b = 1$ $(=m-1,\ m=2)$. Therefore, if u is bounded of order 1, then u is holomorphically continuable. On the other hand (9) has a solution $u(z) = z_1 \log z_1 - z_1$, which is bounded of order $1-\varepsilon$ for any $\varepsilon > 0$ in $\Omega_- = \{z : \operatorname{Re} z_1 < 0\}$ and not continuable across $\partial\Omega_-$ at the origin.

Let us put $\tilde{\Lambda} = \{\alpha \in (\boldsymbol{Z}_+)^n : |\alpha| \leqq m\}$. We next consider the following general nonlinear equation of order m:

$$F(z, (D^\alpha u)) = 0, \tag{10}$$

where $(D^\alpha u) = (D^\alpha u)_{\alpha \in \tilde{\Lambda}}$ and $F(z, \tilde{p})$ is a holomorphic function on $\Omega \times \boldsymbol{C}^{\tilde{N}}$ with $\tilde{p} = (p_\alpha)_{\alpha \in \tilde{\Lambda}}$ and $\tilde{N} = \sharp \tilde{\Lambda}$. Suppose that $\partial\Omega_-$ is non-characteristic with respect to (10), that is,

$$\sum_{|\alpha|=m} \frac{\partial F}{\partial p_\alpha}(z, \tilde{p})(\zeta^0)^\alpha \neq 0 \qquad \text{for } (z, \tilde{p}) \in \Omega \times \boldsymbol{C}^{\tilde{N}}. \tag{A$'$}$$

Differentiating (10) with respect to $\sum_{i=1}^n \zeta_i^0 z_i$, we obtain a quasilinear equation of order $m+1$:

$$\sum_\alpha \frac{\partial F}{\partial p_\alpha}(z, (D^\alpha u)) \sum_{i=1}^n \zeta_i^0 D_i D^\alpha u + \sum_{i=1}^n \zeta_i^0 \frac{\partial F}{\partial z_i}(z, (D^\alpha u)) = 0, \tag{11}$$

for which $\partial\Omega_-$ is non-characteristic. Therefore applying Theorem to (11), we have

Corollary. *If a holomorphic solution u of* (10) *in Ω_- is bounded of order m, then u is holomorphic in a neighborhood of the origin of $\boldsymbol{C}^n$.*

References

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[2] T. Kobayashi: Propagation of singularities for a first order semi-linear system in $\boldsymbol{C}^{n+1}$ (preprint).

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