109. On the Power Semigroup of the Group of Integers

By Takayuki TAMURA
Department of Mathematics, University of California,
Davis, CA 95616 U.S.A.

(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1984)

If $G(\cdot)$ is a group, the power semigroup $\mathcal{P}(G)$ is the semigroup of all nonempty subsets of G with respect to the operation defined by $AB=\{ab : a \in A,\ b \in B\}$ for all $A, B \in \mathcal{P}(G)$. The author and Shafer [5] obtained the group of units of $\mathcal{P}(G)$, and Putcha [4] studied the greatest semilattice decomposition of $\mathcal{P}(G)$ of a finite group G, but we know little about archimedean components of $\mathcal{P}(G)$ of an infinite group G.

Let Z be the group of integers under addition and Z_+ the subsemigroup of positive integers. The operation in $\mathcal{P}(Z)$ is denoted by $X+Y=\{x+y : x \in X,\ y \in Y\}$. For $X \in \mathcal{P}(Z)$ and $m \in Z_+$, we let $mX = \underbrace{X+\cdots+X}_{m}$ and $[a, b]=\{z \in Z : a \leq z \leq b\}$ if $a, b \in Z$ with $a \leq b$. For undefined terminology and basic information on commutative semigroups, the reader should refer to [1], [3].

Let $\mathcal{P}^*(Z)$ denote the subsemigroup of $\mathcal{P}(Z)$ consisting of all finite nonempty subsets of Z. If $X \in \mathcal{P}^*(Z)$, the archimedean component of $\mathcal{P}(Z)$ containing X coincides with that of $\mathcal{P}^*(Z)$ containing X. Let $\mathcal{A}\{0,1\}$ denote the archimedean component of $\mathcal{P}(Z)$ containing the element $\{0,1\}$. The purpose of this paper is to investigate the structure of $\mathcal{A}\{0,1\}$.

Let $X=\{x_1, x_2, \cdots, x_k\} \in \mathcal{P}^*(Z)$ where $x_1 < x_2 < \cdots < x_k$. We define $\min X = x_1$, $\max X = x_k$, $\mathrm{id}(X)=x_2-x_1$, $\mathrm{fd}(X)=x_k-x_{k-1}$, and $\mathrm{md}(X)=\max\{x_2-x_1, \cdots, x_k-x_{k-1}\}$. Note $\mathrm{md}(X) \geqq 1$ unless X is a singleton. If $\mathrm{md}(X)=1$, i.e. $X=[x_1, x_k]$, then X is called *consecutive*. If $\mathrm{id}(X)=\mathrm{fd}(X)=1$, X is called *semi-consecutive*. The following is a main theorem in this paper.

Theorem 1. *Let $X \in \mathcal{P}(Z)$. The following are equivalent:*

(1.1) $X \in \mathcal{A}\{0,1\}$.

(1.2) $nX=\{0,1\}+Y$ *for some* $n \in Z_+$ *and some* $Y \in \mathcal{P}(Z)$.

(1.3) $nX=m\{0,1\}+b$ *for some* $n, m \in Z_+$ *and some* $b \in Z$.

(1.4) X *is semi-consecutive.*

(1.5) nX *is consecutive for some* $n \in Z_+$.

Proof. (1.1)→(1.2) is obvious by archimedeaness.

(1.2)→(1.4). If $X=\{x_1, x_2, \cdots, x_k\}$, $\min(nX)=nx_1$ and the second element of nX is $(n-1)x_1+x_2$. This implies $\mathrm{id}(nX)=\mathrm{id}(X)$. Similarly

$\mathrm{fd}(nX)=\mathrm{fd}(X)$. Since $\{0,1\}+Y$ is semi-consecutive, we have $\mathrm{id}(X)=\mathrm{fd}(X)=1$.

(1.4)→(1.5). First the following lemma is obvious:

Lemma 1.6. *Let $V, W \in \mathcal{P}^*(Z)$ and assume $[a,b]\cap V\neq\phi$. If $V\subseteq W$, then* $\mathrm{md}([a,b]\cap W)\leq \mathrm{md}([a,b]\cap V)$.

To prove "(1.4)→(1.5)" it suffices to prove the following by induction on l.

Lemma 1.7. *Let $n=\mathrm{md}(X)$. If X is semi-consecutive, then* $\mathrm{md}(lX)\leq n-l+1$ *for each l with $1\leq l\leq n$.*

If $l=1$, it is obvious. Assume $l>1$ and Lemma 1.7 holds for l. Let $X=\{0,1,x_2,\cdots,x_{k-1},x_k\}$, $0<1<x_2<\cdots<x_{k-1}<x_k$ and $x_k-x_{k-1}=1$, and let $(l+1)X=D_1\cup D_2$ where $D_1=[0, lx_k+1]\cap(l+1)X$, $D_2=[lx_k, (l+1)x_k]\cap(l+1)X$. Now $lX+\{0,1\}\subset D_1$ and $\mathrm{md}(lX+\{0,1\})\leq n-l$ since $\mathrm{md}(lX)\leq n-l+1$ by induction hypothesis. By Lemma 1.6 we have

$$\mathrm{md}(D_1)\leq \mathrm{md}(lX+\{0,1\})\leq n-l.$$

Next we want to show $\mathrm{md}(D_2)\leq n-l$. The subset lX contains a consecutive subset $C=[lx_{k-1}, lx_k]=\{c_0, c_1,\cdots,c_l\}$ where

$$c_0=lx_{k-1},\cdots,c_i=(l-i)x_{k-1}+ix_k,\cdots,c_l=lx_k.$$

Let $K=[lx_k,(l+1)x_k]$ and $C_0=\{c_l\}$, $C_i=[c_{l-i},c_l]$, $i=1,\cdots,l-1$, $C_l=C$. By induction on i, $\mathrm{md}(K\cap(C_i+X))=\mathrm{md}(K\cap(C_{i-1}+X))-1$, $i=1,\cdots,l$. Since $\mathrm{md}(C_0+X)=n$, $\mathrm{md}(K\cap(C+X))=n-l$. By Lemma 1.6

$$\mathrm{md}(D_2)\leq \mathrm{md}(K\cap(C+X))=n-l$$

because $C+X\subset(l+1)X$. Combining $\mathrm{md}(D_1)\leq n-l$ with $\mathrm{md}(D_2)\leq n-l$, we have $\mathrm{md}((l+1)X)\leq n-l$. Hence Lemma 1.7 holds for all l with $1\leq l\leq n$. In particular, let $l=n$ in Lemma 1.7, then $\mathrm{md}(nX)\leqq 1$. Since nX is not a singleton, $\mathrm{md}(nX)=1$.

(1.5)→(1.3). Since nX is consecutive, there is $b\in Z$ such that $nX-b=[0,m]=m\{0,1\}$ for some $m\in Z_+$.

(1.3)→(1.1). Straightforward.

By Theorem 1, $\{0,1\}+Y\in\mathcal{A}\{0,1\}$ for all $Y\in\mathcal{P}^*(Z)$, so that $\mathcal{A}\{0,1\}$ is an ideal of $\mathcal{P}^*(\mathrm{Z})$. By using the results of [2] and [6] we can describe the structure of $\mathcal{A}\{0,1\}$.

(2) *$\mathcal{A}\{0,1\}$ is homomorphic onto the group Z under $h: X\mapsto$* $\min(X)$.

(3) *Let $X, Y\in\mathcal{A}\{0,1\}$. Then $m\{0,1\}+X=n\{0,1\}+Y$ for some $m,n\in \mathrm{Z}_+$ if and only if* $\min(X)=\min(Y)$.

Let $\mathcal{A}_z=\{X\in\mathcal{A}\{0,1\}: h(X)=z\}$. Then $\mathcal{A}\{0,1\}=\bigcup_{z\in Z}\mathcal{A}_z$, in particular $\mathcal{A}_0$ is a subsemigroup. Define a partial order $\prec$ on each $\mathcal{A}_z$ as follows:

$X, Y\in\mathcal{A}_z$, $X\prec Y$ iff $X=m\{0,1\}+Y$ for some $m\in \mathrm{Z}_+^0=\mathrm{Z}_+\cup\{0\}$.

(4) *$\mathcal{A}_z(\prec)$ forms a tree for each $z\in\mathrm{Z}$, and $\mathcal{A}_0(\prec)$ is order-isomorphic onto $\mathcal{A}_z(\prec)$ for every $z\in\mathrm{Z}$ under $X\mapsto X+z$.*

Theorem 5. *$\mathcal{A}\{0,1\}$ is isomorphic to the direct product of the idempotent-free power joined semigroup $\mathcal{A}_0$ and the group Z.*

Every element X of $\mathcal{A}\{0,1\}$ has the form $X=\bigcup_{i=1}^{l} X_i$, $l\geqq 1$, where each X_i is consecutive, $|X_1|$*) ≥ 2, $|X_l|\geq 2$ and if $l>1$, $X_i\cap X_j=\phi\ (i\neq j)$; $x<y$ for all $x\in X_i$, $y\in X_j$ with $i<j$. Let $X\in\mathcal{A}\{0,1\}$. Then $\{0,1\}\,|\,X$ in $\mathcal{A}\{0,1\}$ if and only if (i) $|X_1|\geq 3$ and $|X_l|\geq 3$, and (ii) if $l>2$, $|X_i|\geq 2$ for all i with $i\neq 1$, $i\neq l$.

Theorem 6. *$\mathcal{A}_0$ consists of $\{0,1\}$, $\{0,1,2\}$, $\{0,1,2,3\}$ and $\{0,1\}\cup Y \cup\{i-1,i\}$ where $i\geq 4$ and Y is any subset of $[2,i-2]$, Y may be empty. If X is not consecutive, $n\{0,1\}+X$ is consecutive for some $n\in \mathrm{Z}_+$ where the least n is $(\mathrm{md}(X))-1$. The homomorphism $h_c:\mathcal{A}_0\to \mathrm{Z}_+$ defined by $h_c(X)=\max(X)$ is the greatest cancellative homomorphism of $\mathcal{A}_0$.*

Theorem 7. *Let C be the set of all consecutive elements of $\mathcal{A}\{0,1\}$. Then C is a subsemigroup of $\mathcal{A}\{0,1\}$, and C is also the greatest cancellative homomorphic image of $\mathcal{A}\{0,1\}$, that is, $C\cong\mathcal{A}\{0,1\}/\rho_1$ where ρ_1 is defined by $X\rho_1 Y$ iff $\max(X)=\max(Y)$. C is a cancellative idempotent-free archimedean semigroup and C is isomorphic to the direct product of Z_+ and Z.*

References

[1] Clifford, A. H. and G. B. Preston: The algebraic theory of semigroups. Math. Surveys No. 7, vol. 1, Amer. Math. Soc., Providence, R. I. (1961).
[2] Levin, R. G. and T. Tamura: Note on commutative power joined semigroups. Pacific J. Math., **35**, 673–680 (1970).
[3] Petrich, M.: Introduction to semigroups. Merrill Publ. Co., Columbus, Ohio (1973).
[4] Putcha, M. S.: On the maximal semilattice decomposition of the power semigroup of a semigroup. Semigroup Forum, **15**, 263–267 (1978).
[5] Tamura, T. and J. Shafer: Power semigroups. Math. Japonicae, **12**, 25–32 (1967).
[6] Tamura, T.: Construction of trees and commutative archimedean semigroups. Mathematische Nachrichten, **36**, 255–287 (1968).

*) $|X_1|$ denotes the number of elements of X_1.