108. On Semi-Free Unitary S^1-Manifolds

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1. Introduction. In [5] C. Kosniowski studied unitary manifolds with an almost effective unitary S^1-action for which the fixed point set is a homology sphere. In this paper we are concerned with unitary S^1-manifolds for which each isotropy group is $\{1\}$ or S^1. These S^1-manifolds are called *semi-free unitary S^1-manifolds*. Let Σ^n be a smooth manifold of the integral homology type of the standard n-sphere S^n. Given an integer k, a homology sphere Σ^{2n} can be equipped with a stable complex structure such that $\tau'(\Sigma^{2n})-\dim_C \tau'=2k\sigma$ if $n\equiv 1 \bmod 4$, and $\tau'(\Sigma^{2n})-\dim_C \tau'=k\sigma$ if $n\equiv 3 \bmod 4$, in the complex K-group $\tilde{K}(\Sigma^{2n})\cong Z(\sigma)$, where τ' is the Whitney sum of the tangent bundle $\tau(\Sigma^{2n})$ and a suitable trivial bundle. The stable complex structures of other spheres are trivial. Let $\Sigma^{2n}(k)$ be the $2n$-sphere with $\tau'(\Sigma^{2n})-\dim_C \tau' =k\sigma$, $n>0$. We then have

Theorem 1. *There exists a semi-free unitary S^1-manifold (M, ϕ) with the fixed point set $\Sigma^{2m}(k_1)+\Sigma^{2m}(k_2)+\cdots+\Sigma^{2m}(k_u)$ such that the normal bundle ν_i of $\Sigma^{2m}(k_i)$ has the m-th Chern class $c^m(\nu_i)=\lambda_i[\Sigma^{2m}(k_i)]$, where $[M]$ indicates the fundamental class of M, if and only if $\sum_{i=1}^u \lambda_i =0$ and $\sum_{i=1}^u k_i=0$.*

The next corollary results from Theorem 1 and the fact that the class $[M]$ of a semi-free S^1-manifold (M, ϕ) in the unitary cobordism group is given by $[M]=\sum_{i=1}^s [P(\nu_i\oplus 1)]$, where the summation is extended over the components $\{F_i\}$ of the fixed point set and $P(\nu_i\oplus 1)$ indicates the projective bundle of $\nu_i\oplus 1$, ν_i the normal bundle of F_i.

Corollary 2. *Let (M, ϕ) be a semi-free unitary S^1-manifold. If the fixed point set is a homology sphere, then M is a boundary.*

The bordism group $F_*^U(S^1)$ of free unitary S^1-manifolds is the free U_*-module with the base $\{[S^{2n+1}, \phi_n]; n=0, 1, 2, \cdots, \phi_n : S^1\times S^{2n+1}\to S^{2n+1}$ is an S^1-action given by $\phi_n(z, v)=zv\}$. By combining the discussion of Theorem 1 with the formal group law theory, we obtain

Theorem 3. *If $\xi\to\Sigma^2$ is an n-dimensional complex vector bundle with $\tau'(\Sigma^2)-\dim_C \tau'=2k\sigma$ in $\tilde{K}(\Sigma^2)\cong Z(\sigma)$, and $c^1(\xi)=\lambda[\Sigma^2]$, then the bordism class $[S(\xi), \phi_\xi]$ of an S^1-action given by $\phi_\xi : S^1\times S(\xi)\to S(\xi)$, $\phi_\xi(z, v)=zv$ is described as follows:*

$$[S(\xi), \phi_\xi]=-\lambda[S^{2n+1}, \phi_n]+(k+\lambda)[CP^1][S^{2n-1}, \phi_{n-1}]$$

$$-\lambda a_{2,1}[S^{2n-3}, \phi_{n-2}] - \cdots - \lambda a_{n,1}[S^1, \phi_0],$$

where $a_{i,j}$ *is the coefficient of the universal formal group law* $F(x, y) = \sum a_{i,j} x^i y^j$ *over the unitary cobordism ring* U^*.

In §2 we prove Theorem 8 which is the key lemma in deriving Theorem 1. In §3 Theorem 3 is proved.

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2. An S^1-action on a sphere bundle over a homology sphere. Let S^1 act on the sphere bundle $S(\xi)$ of an n-dimensional complex vector bundle ξ over Σ^{2m} by the scalar multiplication $\phi_\xi : S^1 \times S(\xi) \to S(\xi)$. Denote by η_P the canonical complex line bundle over the projective bundle $P(\xi)$. The tangent bundle $\tau(P(\xi))$ is stably equivalent to $\pi^!\tau(\Sigma^{2m}) \oplus \bar{\eta}_P \otimes \pi^! \xi$, where $\pi : P(\xi) \to \Sigma^{2m}$ denotes the projection. The Leray-Hirsch Theorem implies

Proposition 4. $H^*(P(\xi); Z)$ *is the free* $H^*(\Sigma^{2m}; Z)$ *module with the base* $\{1, y, y^2, \cdots, y^{n-1}\}$, $y = c^1(\bar{\eta}_P)$, *which satisfies*

$$y^n + \pi^* c^m(\xi) y^{n-m} = 0.$$

Let ξ_n be the Hopf bundle over the complex projective space $CP^n = S^{2n+1}/S^1$ with the first Chern class $x_n = c^1(\xi_n) \in H^2(CP^n; Z)$. The bordism class $[f_\xi] \in U_*(CP^N)$ of a classifying map $f_\xi : P(\xi) \to CP^N$ for the line bundle η_P corresponds to the bordism class of the free unitary S^1-manifold $(S(\xi), \phi_\xi)$. Suppose that $c^m(\xi) = \lambda[\Sigma^{2m}]$, $\lambda \in Z$. The Gysin homomorphism $f_{\xi!} : H^*(P(\xi); Z) \to H^*(CP^N; Z)$ satisfies

$$(5) \qquad \begin{cases} f_{\xi!}(y^l) = -\lambda x_N^{l+N-m-n+1} \\ f_{\xi!}(y^l \pi^*[\Sigma^{2m}]) = x_N^{l+N-n+1}. \end{cases}$$

Let $c_t : \mathrm{Vect}_C(X) \to H^*(X; Z)[[t_1, t_2, \cdots]]$ be the Conner-Floyd characteristic class (cf. [1]); it has the property that

$$c_t(\eta) = 1 + c^1(\eta)t_1 + \{c^1(\eta)\}^2 t_2 + \cdots + \{c^1(\eta)\}^n t_n + \cdots$$

for a line bundle η. The Boardman map $\beta : U^*(X) \to H^*(X)[[t_1, t_2, \cdots]]$ is the multiplicative natural transformation characterized by

$$\beta(c_U^1(\eta)) = c^1(\eta) c_t(\eta) \qquad \text{for a line bundle } \eta,$$

where c_U^i indicates the i-th unitary cobordism Chern class. A map $f : M \to N$ between closed unitary manifolds induces the Gysin homomorphism $f_! : U^*(M) \to U^*(N)$. Let $D : U^*(N) \cong U_*(N)$ be the Atiyah-Thom-Poincaré duality. Then we have

$$(6) \qquad \beta f_!(1) = f_! c_t(\nu_f) \quad \text{and} \quad f_!(1) = D^{-1}[f : M \to N] \quad \text{(cf. [4])},$$

where ν_f indicates the virtual normal bundle of $f : M \to N$. It follows by virtue of splitting principle of a vector bundle that if $\xi \to \Sigma^{2m}$ is an n-dimensional complex vector bundle with $\tau'(\Sigma^{2m}) - \dim_C \tau' = k\sigma$ in $\tilde{K}(\Sigma^{2m}) \cong Z(\sigma)$, then

$$(7) \qquad \begin{cases} c_t(\pi^! \xi \otimes \bar{\eta}_P) = \dfrac{(-1)^{m+1}}{(m-1)!} \pi^* c^m(\xi) \{g(y)\}^{n-1} g^{(m)}(y) + \{g(y)\}^n \\ \pi^* c_t(\tau'(\Sigma^{2m})) = 1 + m!\, k \pi^* [\Sigma^{2m}] t_m \end{cases}$$

where $g(x)=1+t_1x+t_2x^2+\cdots+t_nx^n+\cdots$ and $g^{(m)}(x)$ is the m-th derived function. Assume that the bundle $\xi\to\Sigma^{2m}$ has the m-th Chern class $c^m(\xi)=\lambda[\Sigma^{2m}]$, $\lambda\in Z$. (5), (6) and (7) imply

Theorem 8. *Let $f_\xi: P(\xi)\to CP^N$ be a classifying map for the canonical line bundle η_P over the projective bundle $P(\xi)$. Then*

$$\begin{aligned}\beta D^{-1}&[f_\xi: P(\xi)\to CP^N]\\ &=-\lambda\{g(x_N)\}^{N-n+1}x_N^{N-n-m+1}-m!\,k\{g(x_N)\}^{N-n+1}x_N^{N-n+1}t_m\\ &\quad-\frac{(-1)^{m+1}}{(m-1)!}\lambda\{g(x_N)\}^{N-n}g^{(m)}(x_N)x_N^{N-n+1}.\end{aligned}$$

Proof of Theorem 1. There exists an exact sequence

$$0\longrightarrow SF^U_{2k}(S^1)\xrightarrow{j_*}\sum_{s+t=k}\mathcal{M}^U_{2s,2t}(S^1)\xrightarrow{\partial}F^U_{2k-1}(S^1)\longrightarrow 0\quad([2],[6])$$

where $SF^U_*(S^1)$, $\mathcal{M}^U_{2s,2t}(S^1)$ and $F^U_*(S^1)$ indicate the bordism groups of semi-free unitary S^1-manifolds, complex t-bundles over unitary $2s$-manifolds and free unitary S^1-manifolds, respectively. In the exact sequence $j_*[M,\phi]$ is the sum of bordism classes of normal bundles over the fixed point set and $\partial[\xi\to N]=[S(\xi),\phi_\xi]$. $F^U_{2k-1}(S^1)$ is isomorphic to $U_{2(k-1)}(BU(1))$ which is isomorphic to $U_{2(k-1)}(CP^N)$ for a sufficiently large N. Theorem 1 follows from Theorem 8 and the fact that β is injective.

3. An S^1-action on a sphere bundle over Σ^2. The universal formal group law $F(x,y)=\sum a_{i,j}x^iy^j$ is the formal power series induced from

$$c^1_U(\xi_\infty\hat{\otimes}\xi_\infty)=\sum a_{i,j}(x^U)^i(y^U)^j$$

in $U^*(CP^\infty\times CP^\infty)$, where ξ_∞ the Hopf bundle over CP^∞, $x^U=c^1_U(\xi_\infty\hat{\otimes}1)$ and $y^U=c^1_U(1\hat{\otimes}\xi_\infty)$. Let $h(x)=xg(x)$. The Boardman map induces an isomorphism $\beta_Q: U^*(CP^\infty\times CP^\infty)\otimes Q\cong H^*(CP^\infty\times CP^\infty;Q)[[t_1,t_2,\cdots]]$. Then we have

$$\beta_Q^{-1}h^{-1}(\beta F(x^U,y^U))=\beta_Q^{-1}h^{-1}(\beta(x^U))+\beta_Q^{-1}h^{-1}(\beta(y^U)).$$

This means that $\beta_Q^{-1}h^{-1}\beta$ is the logarithm and

$$\beta_Q^{-1}h^{-1}\beta(\tilde{x}^U)=\tilde{x}^U+\frac{[CP^1]}{2}(\tilde{x}^U)^2+\cdots+\frac{[CP^k]}{k+1}(\tilde{x}^U)^{k+1}+\cdots$$

where $\tilde{x}^U=c^1_U(\xi_\infty)$. Letting $x^H=c^1(\xi_\infty)$, it follows that

$$x^H=h(x^H)+\frac{\beta[CP^1]}{2}\{h(x^H)\}^2+\cdots+\frac{\beta[CP^k]}{k+1}\{h(x^H)\}^{k+1}+\cdots.$$

We then obtain

$$h'(x^H)\{1+\beta[CP^1]h(x^H)+\beta[CP^2]\{h(x^H)\}^2+\cdots+\beta[CP^n]\{h(x^H)\}^n+\cdots\}=1.$$

Noting that

$$\{1+[CP^1]\tilde{x}^U+[CP^2](\tilde{x}^U)^2+\cdots\}\{1+a_{1,1}\tilde{x}^U+a_{2,1}(\tilde{x}^U)^2+\cdots\}=1$$

(cf. [3]), we can show that

$$(9)\qquad x^Hg'(x^H)+g(x^H)=\beta\{1+a_{1,1}\tilde{x}^U+a_{2,1}(\tilde{x}^U)^2+\cdots\}.$$

Since the Atiyah-Thom-Poincaré isomorphism sends $(x^U_N)^k$ to $[CP^{N-k}\subset CP^N]$, $x^U_N=c^1_U(\xi_N)$, Theorem 3 follows from Theorem 8 and (9).

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