99. *On a Semilinear Diffusion Equation on a Riemannian Manifold and its Stable Equilibrium Solutions*

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§ 1. **Introduction.** Let (M, g) be a connected orientable compact C^∞ Riemannian manifold with (possibly empty) smooth boundary ∂M.

We consider the following semilinear diffusion equation and its equilibrium solutions.

$$(1.1) \qquad \frac{\partial u}{\partial t} = \Delta u + f(u) \qquad \text{in } (0, \infty)\times M$$

$$(1.2) \qquad \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } (0, \infty)\times \partial M$$

where f is a smooth function on $\boldsymbol{R}$ into $\boldsymbol{R}$, $\Delta=\text{div grad}$ is the Laplace-Beltrami operator with respect to the metric g and ν denotes the outward unit normal vector on ∂M. In the case $\partial M=\phi$, we eliminate (1.2).

In this note, we will report that the system (1.1)–(1.2) does not admit any spatially inhomogeneous stable equilibrium solution under some geometrical assumptions for M, while it is not the case with some (M, g) and f.

In the case that M is a bounded domain in the Euclidean space, Matano has proved in [4] that if the domain is convex, then any stable equilibrium solution must be a constant function, and he has also constructed a domain and a function f for which the system (1.1)–(1.2) admits a non-constant stable equilibrium solution. Then our result may be regarded as a generalization of his result to the case of manifolds.

§ 2. **Statement of the results.**

Theorem 1. *Assume the following conditions* (1) *and* (2):

(1) *M has non-negative Ricci curvature,* i.e. *for any $x \in M$ and $X \in T_xM$, $R(X, X)\geqq 0$ holds. Here $R(\cdot, \cdot)$ denotes the Ricci tensor.*

(2) *The second fundamental form of ∂M with respect to ν in M is non-positive definite.*

Then any non-constant equilibrium solution of (1.1)–(1.2) *is unstable.*

Remark 1. In the case $\partial M=\phi$, we eliminate the assumption (2) in Theorem 1.

Remark 2. If M is a bounded subdomain of $\boldsymbol{R}^n$ with smooth boundary, the assumption (2) is equivalent to the convexity of M.

§3. **Outline of the proof of Theorem 1.**

To show Theorem 1, we prepare an inequality to estimate the first eigenvalue of the linearized operator.

Proposition. *Let u be an equilibrium solution of* (1.1)–(1.2). *Then we have the following inequality*:

$$\mathcal{H}_u(|\mathrm{grad}\, u|)+\int_M R(\mathrm{grad}\, u, \mathrm{grad}\, u)\, dx$$
$$-\int_{\partial M} |\mathrm{grad}\, u| \frac{\partial}{\partial \nu} |\mathrm{grad}\, u|\, dS \leqq 0$$

where $\mathcal{H}_u(v) \equiv \int_M \{|\mathrm{grad}\, v|^2 - f'(u)v^2\}\, dx$ for $v \in H^1(M)$.

This proposition is proved by localization and integration by the aid of the following lemma.

Lemma. *For any domain $\Omega \subset M$ and any $\psi \in C^3(\Omega)$ such that* $\mathrm{grad}\, \psi \neq 0$ *in Ω, we have the following inequality*:

$$\mathrm{grad}\, \psi(\Delta\psi) - |\mathrm{grad}\, \psi|\, \Delta(|\mathrm{grad}\, \psi|) + R(\mathrm{grad}\, \psi, \mathrm{grad}\, \psi) \leqq 0 \quad \text{in } \Omega.$$

We will sketch the proof of Theorem 1. We have only to show that the first eigenvalue λ_1 of the operator $\Delta + f'(u)\cdot$ with Neumann boundary condition is positive when u is a non-constant equilibrium solution. By the characterization of the eigenvalue, we have $-\lambda_1 = \inf_{\psi \in H^1(M)} \mathcal{H}_u(\psi)/\|\psi\|^2_{L^2(M)}$. From the assumption of Theorem 1 and by Proposition, we can prove $-\lambda_1 \leqq 0$. If we assume $\lambda_1 = 0$, then $v \equiv |\mathrm{grad}\, u|$ must be the first eigenfunction for the Neumann boundary value problem and accordingly v has definite sign in M and up to ∂M. Therefore u attains its maximum on ∂M. But u satisfies the Neumann boundary condition and so we have $\mathrm{grad}_{\partial M}(u|_{\partial M}) = (\mathrm{grad}\, u)|_{\partial M}$. Here $\mathrm{grad}_{\partial M}$ is the gradient operator in the compact Riemannian manifold $(\partial M, g|_{\partial M})$. Hence $v = |\mathrm{grad}\, u|$ must vanish on some point of ∂M. Thus we have a contradiction and we have shown that λ_1 is positive.

§4. **Manifold admitting non-constant stable solutions.**

In this section, we will construct a manifold and a function f for which the equation (1.1) admits a non-constant stable equilibrium solution.

Let (M_i, g_i), $1 \leqq i \leqq m$, be n-dimensional connected compact orientable C^∞ Riemannian manifolds without boundary.

For each i $(1 \leqq i \leqq m)$, we fix $m-1$ points $P_{i,1}, \cdots, P_{i,m-1} \in M_i$ and define for $\zeta > 0$,

$B_{i,j}(\zeta) \equiv$ open geodesic ball of radius ζ about $P_{i,j}$

$M_i(\zeta) \equiv M_i - \bigcup_{j=1}^{m-1} \overline{B_{i,j}(\zeta)}$

$S_\zeta \equiv (n-1)$-sphere of radius ζ in $\boldsymbol{R}^n$.

Let (M_ζ, g_ζ) be a connected compact orientable C^∞ Riemannian manifold which has no boundary and satisfies the following conditions (M.1), (M.2), (M.3) and (M.4):

(M.1) For each i $(1\leqq i\leqq m)$, $(M_i(\zeta), g_i)$ can be isometrically imbedded in (M_ζ, g_ζ) in such a way that $\iota_i(M_i(\zeta))\cap\iota_j(M_j(\zeta))=\phi$ for any i and j $(1\leqq i<j\leqq m)$. Here ι_i is the imbedding mapping of $M_i(\zeta)$ into M_ζ.

(M.2) $Q(\zeta)\equiv M_\zeta-\bigcup_{i=1}^m \iota_i(M_i(\zeta))$ is diffeomorphic to
$$([-1,1]\times S_1)\cup([-1,1]\times S_1)\cup\cdots\cup([-1,1]\times S_1)$$
which is the union of mutually disjoint $m(m-1)/2$ cylindrical hypersurfaces.

(M.3) For some $\rho>0$, the cylinder $(-\rho,\rho)\times S_\zeta$ can be isometrically imbedded in any connected component of $Q(\zeta)$.

(M.4) $\lim_{\zeta\to 0}\mathrm{Vol}\,(Q(\zeta))=0$.

Next we determine the nonlinear term f.

(f) f is a real valued smooth function on $\boldsymbol{R}$ and there are m distinct points $a_1, a_2, \cdots, a_m\in\boldsymbol{R}$ such that $f(a_i)=0$ and $f'(a_i)<0$ hold for any i $(1\leqq i\leqq m)$.

We consider in (M_ζ, g_ζ) the equation (1.1) for f which we have constructed above. Then we have the following theorem.

Theorem 2. *Under the assumptions* (M.1), (M.2), (M.3), (M.4) *and* (f), *there is a stable equilibrium solution u_ζ of* (1.1) *in (M_ζ, g_ζ) which satisfies the following properties.*

$$\lim_{\zeta\to 0}\|u_\zeta-a_i\|_{L^2(\iota_i(M_i(\zeta)))}=0 \qquad (1\leqq i\leqq m)$$
$$\lim_{\zeta\to 0} u_\zeta=a_i \text{ in } C^\infty(\iota_i(M_i(\eta))) \text{ for any small } \eta>0\ (1\leqq i\leqq m).$$

Remark 3. Theorem 2 may be regarded as an analogue to Theorem 6.2, Corollary 6.3 and Remark 6.4 in [4]. But our situation concerning f and (M_ζ, g_ζ) is more general than that of [4].

For the proof of Theorem 2, it is a device to use the following inequality:
$$\frac{1}{\lambda_{q+1}}\int_D|\mathrm{grad}\,\psi|^2dx+\sum_{k=1}^q\frac{\lambda_{q+1}-\lambda_k}{\lambda_{q+1}}\Bigl(\int_D\psi\cdot\psi_k\,dx\Bigr)^2\geqq\int_D|\psi|^2dx$$
for any $\psi\in H^1(D)$ and any $q\geqq 0$. Here D is a connected compact orientable C^∞ Riemannian manifold with smooth boundary and $\{\lambda_q\}_{q=1}^\infty$ and $\{\psi_q\}_{q=1}^\infty$ are respectively the sequence of eigenvalues arranged in increasing order and the complete system of the corresponding orthonormalized eigenfunctions associated with $-\Delta$ with Neumann boundary condition. This inequality is easily proved by eigenfunction expansion.

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