98. A New Proof of the Schiffer's Identities on Planar Riemann Surfaces*)

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Introduction. Let D be a domain in the complex plane, bounded by a finite number of curves. For a point z_0 in D, there exist two slit functions $P_j(z)$ $(j=0,1)$ on D such that (i) $P_j(z)-(z-z_0)^{-1}$ are regular and vanish at z_0, (ii) $P_0(z)(P_1(z))$ maps D conformally to a domain whose boundary consists of horizontal (vertical) slits. Setting $\Phi_+ =(P_0+P_1)/2$ and $\Phi_-=(P_0-P_1)/2$, Schiffer [9] found that Φ_+ becomes schlicht and the following remarkable identities hold;

$$(*)\qquad \Phi_-(z)=-\frac{1}{\pi}\iint_A \frac{d\xi d\eta}{\zeta-\Phi_+(z)} \qquad \text{for } z\in D,\quad \zeta=\xi+i\eta$$

$$(**)\qquad \iint_A \frac{d\xi d\eta}{(\zeta-w)^2}=0 \qquad \text{for } w\in \mathring{A}$$

where $\mathring{A}$ is the interior of the complement A of the image $\Phi_+(D)$ and the integral in (**) is the Cauchy's principal value. Recently, Burbea showed an application of relation (*) ([2]) and gave another proof of (*) ([3]). He states there without proof that (*) remains valid for general plane domain by using its exhaustion. However, for this proof some detailed studies must be necessary, as the set A varies with the exhaustion.

The main purpose of this note is to give a new direct proof of (*) and (**) for the (generalized) slit functions on an arbitrary planar Riemann surface. Our approach is essentially different from the known one, actually the proof is based on the Hilbert transform and the extremal properties of Φ_+ and Φ_- defined by those slit functions.

1. Preliminaries. For an arbitrary Riemann surface R, let $\Gamma_a=\Gamma_a(R)$ be the Hilbert space of square integrable analytic differentials ω on R with norm $\|\omega\|=\left(\iint_R \omega\wedge *\bar{\omega}\right)^{1/2}$, and $\Gamma_{ase}=\{\omega\in\Gamma_a \mid \omega$ is semiexact, i.e. $\int_\gamma \omega=0$ for every closed curve γ dividing $R\}$. Let q be a point of R and ζ be a fixed local parameter at q with $\zeta(q)=0$. For $\omega\in\Gamma_a$ we denote by $\omega^{(n)}(q)$ the n-th derivative $\omega(\zeta)$ at $\zeta=0$, where $\omega=\omega(\zeta)d\zeta$. Now it is known (cf. [4], [5], also [8], [10]) that for every integer $n\geq 1$ there exist uniquely the *semiexact canonical* (mero-

*) Dedicated to Professor Sigeru Mizohata on his 60th birthday.

morphic) *differentials* ψ_q^n and $\tilde{\psi}_q^n$ on R with a pole at q such that (i) about q,

$$(1)\qquad \psi_q^n = d(\zeta^{-n}+a_1\zeta+\cdots),\quad \tilde{\psi}_q^n = id(\zeta^{-n}+b_1\zeta+\cdots),\quad (i=\sqrt{-1})$$

(ii) $u=\mathrm{Re}\int\psi_q^n$ and $\tilde{u}=\mathrm{Re}\int\tilde{\psi}_q^n$ are single-valued on $R-U$, U being a neighborhood of q, and they have the following property;

$$(2)\qquad (\mathrm{Re}\,\psi_q^n,\theta)_{R-U} = -\int_{\partial U} u^*\bar{\theta},\qquad (\mathrm{Re}\,\tilde{\psi}_q^n,\theta)_{R-U} = -\int_{\partial U}\tilde{u}^*\bar{\theta}$$

is valid for every (complex) harmonic differential θ on $R-U$ with finite norm such that the conjugate differential $^*\theta$ of θ is semiexact, where the line integral is taken in the positive sense with respect to U. Obviously (2) is then valid for every $\theta\in\Gamma_{\mathrm{ase}}(R)$.

We write $\varphi_q^n=-i\tilde{\psi}_q^n$ and $\phi_-^n=(\varphi_q^n-\psi_q^n)/2$, $\phi_+^n=(\varphi_q^n+\psi_q^n)/2$. Then ϕ_+^n has a pole at q of order $n+1$ and belongs to $\Gamma_{\mathrm{ase}}(R-\bar{U})$, while, $\phi_-^n\in\Gamma_{\mathrm{ase}}(R)$ and

$$\phi_-^n = d(s_1\zeta+s_2\zeta^2+\cdots)\qquad \text{at } q(\zeta),$$

where $s_k=(b_k-a_k)/2$ with a_k and b_k in (1). By using (2) one can easily prove that $(\omega,\phi_-^n)=2\pi\omega^{(n-1)}(q)/(n-1)!$ for any $\omega\in\Gamma_{\mathrm{ase}}(R)$, from which we have

Proposition 1. 1) $\|\phi_-^n\|^2=2\pi n s_n$, *and hence* s_n *is real and non-negative.* 2) (*extremal property*) *If* σ *is any differential of* Γ_{ase} *such that* $\sigma^{(n-1)}(q)=(\phi_-^n)^{(n-1)}(q)$ $(=n!\,s_n)$, *then* $\|\phi_-^n\|\le\|\sigma\|$, *where the equality holds if and only if* $\phi_-^n\equiv\sigma$.

If R is planar (i.e. of genus zero), semiexactness implies exactness and we have the following extremal property for ϕ_+^n.

Proposition 2. *Let* q, ζ *and* n *be fixed as above. Let* $\mathcal{F}$ *be the set of meromorphic functions* f *on a planar Riemann surface* R *with a pole at* q *such that* $f(\zeta)=\zeta^{-n}+c_0+c_1\zeta+\cdots$ *at* $q(\zeta)$ *and* $\|df\|_{R-U}<\infty$, U *being a neighborhood of* q. *Then* $\Phi_+^n=\int\phi_+^n\,(\in\mathcal{F})$ *maximizes the integral* $(2i)^{-1}\int_{\partial R} f d\bar{f}$ *for* $f\in\mathcal{F}$ *and the maximum is* $\pi n s_n$, *which is attained if and only if* f *is identical with* Φ_+^n *except a constant, where the integral over* ∂R *means the increasing limit of* $(2i)^{-1}\int_{\partial\Omega} f d\bar{f}$ *as* (*regular region*) $\Omega\nearrow R$.

We give a brief proof. Let U_ε be ε-neighborhood of q, then $(\phi_+^n,\psi_q^n)_{R_\varepsilon}=(\phi_+^n,\varphi_q^n)_{R_\varepsilon}=2\pi n(\varepsilon^{-2n}-s_n)+O(\varepsilon^2)$ with $R_\varepsilon=R-\bar{U}_\varepsilon$. Hence $\|\phi_+^n\|_{R_\varepsilon}^2$ has the same value. While, for any $f\in\mathcal{F}$, $\|df\|_{R_\varepsilon}^2=i\int_{\partial R} f d\bar{f}+2\pi n\varepsilon^{-2n}+O(\varepsilon^2)$. Taking $f=\Phi_+^n$, we have thus two expressions for $\|\phi_+^n\|_{R_\varepsilon}^2$, from which we have $(2i)^{-1}\int_{\partial R}\Phi_+^n d\bar{\Phi}_+^n=\pi n s_n$. Put $h=f-\Phi_+^n\,(\in\Gamma_{\mathrm{ase}})$. Since $(\omega,\phi_+^n)_{R_\varepsilon}\to 0$ as $\varepsilon\to 0$ for any $\omega\in\Gamma_{\mathrm{ase}}$, we have

$(2i)^{-1}\int_{\partial R} f d\bar{f} + (\|dh\|^2)/2 = \pi n s_n$ which implies the assertion.

2. Let R be a planar Riemann surface and q, ζ be as before. We write $P_0 = \int \varphi_q^1$ and $P_1 = \int \psi_q^1$ with the normalization of constant at q;

$$P_0(\zeta) = \zeta^{-1} + b\zeta + \cdots, \quad P_1(\zeta) = \zeta^{-1} + a\zeta + \cdots.$$

These give the (*generalized*) *slit functions* on R. Namely, it is known (cf. [5], [6], also [1]) that $P_0(P_1)$ maps R conformally onto a plane region whose boundary consists of horizontal (vertical) slits respectively, where slits may reduce to points, and that $\Phi_+ = (P_0 + P_1)/2$ $(= \Phi_+^1 + \text{const})$ is univalent on R. In this case, $(2i)^{-1}\int_{\partial R} \Phi_+ d\bar{\Phi}_+$ gives the area of the complement of the image $\Phi_+(R)$ and Proposition 2 implies that

$$(3) \qquad |A| = \pi s, \qquad A = C - \Phi_+(R),$$

where $s = (b-a)/2$ is the *span* of R at $q(\zeta)$ and $|A|$ stands for the 2-dimensional Lebesgue measure of A. To prove the generalization of Schiffer's identities we consider the following integrals over a compact set $E(\subset C)$ of positive measure;

$$T_E(z) = -\frac{1}{\pi}\iint_E \frac{d\xi d\eta}{\zeta - z}, \qquad S_E(z) = -\frac{1}{\pi}\iint_E \frac{d\xi d\eta}{(\zeta - z)^2},$$

where the latter integral is the principal value. Then $T_E(z)$ and $S_E(z)$ are holomorphic outside of E and $T'_E(z) = S_E(z)$ there.

Theorem. *Let R be a planar Riemann surface and $R \notin O_{AD}$. Let P_0 and P_1 be the (generalized) horizontal and vertical slit functions of R respectively and $\Phi_- = (P_0 - P_1)/2$, $\Phi_+ = (P_0 + P_1)/2$. Then the complement A of the image $\Phi_+(R)$ has positive measure and we have*

$$(4) \qquad \Phi_-(p) = T_A \circ \Phi_+(p) \qquad \text{for every point } p \in R$$

$$(5) \qquad S_A(z) = 0 \qquad \text{for every } z \in \mathring{A}$$

and $S_A(z)$ can be represented, except for the points of ∂A, as the Hilbert transform of the characteristic function of A.

Proof. The span s of R is positive if and only if $R \notin O_{AD}$ (cf. [7]). Hence $|A| > 0$ by (3). Note that A is compact. The function $T_A(z)$ is holomorphic in $B = \Phi_+(R)$ and $T'_A(z) = S_A(z)$ for $z \in B$. Let χ_A be the characteristic function of A. Then for every $z \in B$, $S_A(z)$ can be expressed as the Hilbert transform $S\chi_A(z)$ of $\chi_A \in L^2$. (More generally, the Hilbert transform $Sf(z)$ for $f \in L^2$ has the integral representation $-\pi^{-1}\iint f(\zeta)(\zeta - z)^{-2} d\xi d\eta$ for every z on an open set where f is Hölder continuous. I owe to Prof. Mizohata for this matter.) Moreover, since S is an isometry on L^2, it follows that

$$(6) \qquad \begin{aligned} \|dT_A\|_B^2 &= 2\|S_A\|_{L^2(B)}^2 = 2\|S\chi_A\|_{L^2(B)}^2 \\ &= 2(\|S\chi_A\|_{L^2}^2 - \|S\chi_A\|_{L^2(A)}^2) \\ &\le 2\|S\chi_A\|_{L^2}^2 = 2\|\chi_A\|_{L^2}^2 = 2|A|. \end{aligned}$$

Using (3) we know that the expansion of $T_A(z)$ at ∞ is of the form

$$T_A(z)=sz^{-1}+\text{const}\, z^{-2}+\cdots.$$

Hence the composite function $g=T_A\circ\Phi_+$ is holomorphic on R and $g(\zeta)=s\zeta+\cdots$ in terms of local parameter ζ at q. By Proposition 1, (6) and (3) we have hence

$$2\pi s=\|d\Phi_-\|^2\leq\|dg\|^2=\|dT_A\|_B^2\leq 2|A|=2\pi s,$$

and so $\|dT_A\|_B^2=2|A|=2\pi s$ and $\|d\Phi_-\|=\|dg\|$. The latter equality implies $g\equiv\Phi_-$ on account of Proposition 1 and $g(q)=\Phi_-(q)=0$, which shows the relation (4). The former gives the equality in (6), so $\|S\chi_A\|_{L^2(A)}=0$, hence we may assume $S\chi_A(z)=0$ for $z\in A$. Since $S\chi_A(z)$ has the integral representation $S_A(z)$ for $z\in \mathring{A}$, the assertion (5) follows.

References

[1] L. V. Ahlfors and L. Sario: Riemann Surfaces. Princeton Univ. Press, Princeton (1960).

[2] J. Burbea: Capacities and spans on Riemann surfaces. Proc. Amer. Math. Soc., **72**, 327–332 (1978).

[3] ——: The Schwarzian derivative and the Poincaré metric. Pacif. J. of Math., **85**, 345–354 (1979).

[4] Y. Kusunoki: Theory of Abelian integrals and its applications to conformal mappings. Mem. Col. Sci. Kyoto Univ., Ser. A. Math., **32**, 235–258 (1959).

[5] ——: Characterizations of canonical differentials. J. Math. Kyoto Univ., **5**, 197–207 (1966).

[6] M. Mori: Canonical conformal mappings of open Riemann surfaces. ibid., **3**, 169–192 (1964).

[7] A. Pfluger: Theorie der Riemannschen Flächen. Springer-Verlag, Berlin-Göttingen-Heidelberg (1957).

[8] L. Sario and K. Oikawa: Capacity Functions. Springer-Verlag, Berlin-Heidelberg-New York (1969).

[9] M. Schiffer: The span of multiply connected domains. Duke Math. J., **10**, 209–216 (1943).

[10] M. Shiba: Some general properties of behavior spaces of harmonic semiexact differentials on open Riemann surfaces. Hiroshima Math. J., **8**, 151–164 (1978).