9. On Generalized Gelfand Pairs

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1. Introduction. Let G be a unimodular, locally compact second countable (lcsc) group and H a closed unimodular subgroup. Put $S=G/H$ and denote by $D(S)$ the space of Schwartz-Bruhat functions on S, endowed with the usual inductive-limit topology. Let $D'(S)$ be the topological anti-dual of $D(S)$, endowed with the strong topology. Let π be a continuous irreducible unitary representation of G on the Hilbert space $\mathcal{H}$.

We consider the following three problems:

(i) Can π be realized on a Hilbert-subspace of $D'(S)$: does there exist a continuous linear injection $j: \mathcal{H} \to D'(S)$ such that

$$j\pi(x)=L_x j$$

for all $x \in G$ (L_x denotes left translation by x)?

(ii) What can be said about uniqueness (up to scalar multiplication) of such a realization?

(iii) In case of unique realization, does $j(\mathcal{H})$ consist of functions (C^∞, locally L^2)?

Some results on these problems, obtained by L. Schwartz, P. Cartier, J. Faraut, G. E. F. Thomas, F. J. M. Klamer together with the author's results will be reported in this work. Details of the proofs of the author's results will be published elsewhere.

2. Realization on a Hilbert-subspace. We start with a result which can be obtained from [2] and [3]. Let G be a lcsc group, H a closed subgroup of G. For technical reasons, we assume both G and H to be unimodular. Put $S=G/H$. Let π be an irreducible unitary representation of G on the Hilbert space $\mathcal{H}$. The subspace $\mathcal{H}_\infty$ of C^∞-vectors in $\mathcal{H}$ can be endowed with a natural Sobolev-type topology (cf. [2], §1). $\mathcal{H}_\infty$ is G-invariant. Denote $\mathcal{H}_{-\infty}$ the anti-dual of $\mathcal{H}_\infty$. Then $\mathcal{H}_\infty \subset \mathcal{H} \subset \mathcal{H}_{-\infty}$ and the injections are continuous. G acts on $\mathcal{H}_{-\infty}$, the corresponding representation is called $\pi_{-\infty}$. Define

$$\mathcal{H}^H_{-\infty}=\{a \in \mathcal{H}_{-\infty} \mid \pi_{-\infty}(h)a=a \text{ for all } h \in H\}.$$

Then we have the following result.

Theorem 1. *π can be realized on a Hilbert-subspace of $D'(S)$ if and only if $\mathcal{H}^H_{-\infty} \neq (0)$. There is a one-to-one correspondence between the non-zero elements of $\mathcal{H}^H_{-\infty}$ and the continuous linear injections $j: \mathcal{H} \to D'(S)$ satisfying $j\pi(x)=L_x j (x \in G)$. To $a \neq 0$ in $\mathcal{H}^H_{-\infty}$ corresponds*

j, such that $j^*: D(S)\to\mathcal{H}$ is given by $j^*(\phi)=\pi_{-\infty}(\phi)a$.

We now turn to two concrete situations.

a. Let G_0 be a type I, unimodular lcsc group. Put $G=G_0\times G_0$ and $H=\operatorname{diag}(G)$. Then the following result is due to Klamer ([4], Ch. 3, §1).

Theorem 2. *Let π be an irreducible unitary representation of G. π can be realized on a Hilbert-subspace of $D'(G_0)$ if and only if π is of the form $\bar{\pi}_0\otimes\pi_0$ for some irreducible unitary representation π_0 of G_0 and π_0 has a character.*

b. Let G be semi-direct product of the closed subgroup H and the closed normal, abelian subgroup $N: G=N\ltimes H$. Write $\hat{N}$ for the dual group of N. Let us assume:

(i) G is a regular semi-direct product in the Mackey sense, i.e. there exists a Borel section for the H-orbits in $\hat{N}$.

(ii) G and H are unimodular.

(iii) For all $\gamma\in\hat{N}$ the H-orbit O_γ of γ in $\hat{N}$ is open in its closure.

Then one has, due to Klamer ([4], Ch. 4, §1):

Theorem 3. *Let π be an irreducible unitary representation of G. π can be realized on a Hilbert-subspace $\mathcal{H}_\pi$ of $D'(N)$ if and only if there exists $\gamma\in\hat{N}$ such that*

(i) π is equivalent to the representation induced by the unitary representation $\gamma\otimes I$ of $N\ltimes H_\gamma$, where H_γ is the stabilizer of γ in H, for some $\gamma\in\hat{N}$.

(ii) O_γ carries an H-invariant positive measure μ_γ.

(iii) μ_γ is naturally a tempered Radon measure on $\hat{N}$ (tempered in the Bruhat sense). If (i)–(iii) *are satisfied, $\mathcal{H}_\pi=\mathcal{F}(L^2(\mu_\gamma))$, $\mathcal{F}$ denoting Fourier transform.*

Examples will be given in §§3 and 4.

3. Uniqueness. This leads to the notion of a (generalized) Gelfand pair (G,H). Let us first discuss the classical situation where H is compact.

Theorem 4 (Thomas [7]). *Let G be a locally compact, second countable and unimodular group, H a compact subgroup. Put $S=G/H$. Let Γ_G denote the cone of G-invariant Hilbert-subspaces of $D'(S)$, G acting unitarily and write* ext Γ_G *for the set of extremal rays of Γ_G.*

The following assertions are equivalent:

(i) *(G,H) is a Gelfand pair,* i.e. *the convolution algebra $L^1(H\backslash G/H)$ of L^1-functions f on G, satisfying $f(hxh')=f(x)$ ($x\in G$; $h'\in H$) is commutative.*

(ii) *Γ_G is a lattice in its usual order.*

(iii) *The commutant $(G|_{\mathcal{H}})'$ is abelian for every $\mathcal{H}\in\Gamma_G$.*

(iv) *$(G|_{L^2(S)})'$ is abelian.*

(v) *G' is abelian (G to be considered as a subgroup of $GL(D'(S))$) [$G' \simeq \mathcal{E}'$ ($H\, G/H$), the bi-H-invariant distributions on G with compact support].*

(vi) *$\mathcal{H}_1$ and $\mathcal{H}_2$ in* ext (Γ_G) *are either proportional or not equivalent. If in addition G is a connected Lie group,* (i)–(vi) *are equivalent to:*

(vii) *The algebra $\boldsymbol{D}(s)$ of G-invariant differential operators on S is commutative.*

Now let H be any closed (non-neccessarily compact) unimodular subgroup of G. Then (i) has no meaning in general, but (ii), (iii) and (vi) have. Moreover (ii) and (iii) are equivalent and imply (vi). If G is type I, (ii), (iii) and (iv) are equivalent.

So we define:

Definition 5. Let G be a lcsc unimodular group, H a closed unimodular subgroup. The pair (G, H) is called a generalized Gelfand pair if Γ_G is a lattice.

Most examples are related to *symmetric spaces.*

Let G be a Lie group with an involution σ. Let H be a subgroup of G such that $G_\sigma^0 \subset H \subset G_\sigma$, where G_σ is the set of fixed points of σ and G_σ^0 its connected component of the identity element. The pair (G, H) is called a symmetric pair and G/H is a symmetric space.

It is known that in the following cases (G, H) is a generalized Gelfand pair:

(1) $(G \times G$, diag $(G \times G))$, G being a lcsc unimodular group.

(2) (G, N), G being the semi-direct product of a closed unimodular subgroup H and a closed normal abelian subgroup N.

(3) (G, H) is a Riemannian symmetric pair (i.e. $Ad(H)$ is compact).

(4) G is a connected nilpotent Lie group, σ an involution on G and H a subgroup such that $G_\sigma^0 \subset H \subset G_\sigma$ (Y. Benoist [1]).

(5) $G = U(p, q; F)$, $H = U(1) \times U(p-1, q; F)$ where $F = \boldsymbol{R}$, $\boldsymbol{C}$, or the quarternions (Thomas, 1983). The same proof applies to the octave hyperbolic plane.

(6) $G = SL(n, R)$, $H = S(GL(1) \times GL(n-1))$ for $n \geqq 3$. If $n = 2$, (G, H) is *not* a generalized Gelfand pair.

Remark. Let (G, H) be a symmetric pair with G connected. By a well-known result of Lichnerowicz [5], the algebra $\boldsymbol{D}(S)$ of G-invariant differential operators on $S = G/H$ is commutative, provided S carries a G-invariant positive measure. Example 6 shows that in general the commutativity of $\boldsymbol{D}(S)$ does not imply that (G, H) is a generalized Gelfand pair, though G and H are unimodular groups. Compare this fact with Theorem 4, where H is supposed to be compact.

4. Regularity. Let G and H be as usual. Assume that (G, H) is a generalized Gelfand pair. We intend to answer the following question: let $\mathcal{H}$ be an irreducible G-invariant Hilbert-subspace of $D'(S)$. Is $\mathcal{H} \subset L^2_{\text{loc}}(S)$, the space of locally L^2-functions on S with respect to a positive G-invariant measure on S?

(1) Let H be compact. Any irreducible G-invariant Hilbert-subspace of $D'(S)$ is contained in $C^\infty(S)$.

(2) The case: $G \times G$, $H = \text{diag}(G \times G)$, $S = G$ (G unimodular and type I).

Let π be an irreducible unitary representation of G. Assume that the distribution-character of π exists and denote by $\mathcal{H}_\pi$ the irreducible bi-invariant Hilbert-subspace of $D'(G)$ corresponding to $\bar{\pi} \otimes \pi$ (cf. Theorem 2). Call π *regular* if $\mathcal{H}_\pi \subset L^2_{\text{loc}}(G)$.

G is called regular if all irreducible unitary representations π are regular.

a. Compact, abelian and connected semisimple (Lie) groups are regular (Klamer [4], Theorem 3.1.12).

b. Let N_n denote the group of $n \times n$ uppertriangular unipotent matrices. Then we have:

(α) N_3 and N_4 are regular (cf. also [4], Ch. 4).

(β) The representations of N_n, which occur in the Plancherel formula, are regular.

Remark. It is still an open problem whether any connected (and simply connected) nilpotent Lie group is regular or not.

(3) Let $G = N \ltimes H$ as in 2.b., $S = N$. Put $\pi_\gamma = \pi_{\gamma,1}$ ($\gamma \in \hat{N}$). Let O_γ carry an H-invariant positive measure μ_γ and assume μ_γ to be a tempered Radon measure on $\hat{N}$. Then π_γ can be realized on $D'(S)$.

Theorem 6. *The realization of π_γ takes place inside $L^2_{\text{loc}}(N)$ if and only if $\mathcal{F}(L^2(\mu_\gamma)) \subset L^2_{\text{loc}}(N)$.*

Example. Let Q be a non-degenerate quadratic form on V, where $V = \boldsymbol{R}^n$, C^n or k^n (k a local field of characteristic $\neq 2$).

Denote by $O(Q)$ the orthogonal group of Q and let $G = V \ltimes O(Q)$. G and $O(Q)$ satisfy the assumptions of 2.b. Moreover, each O_γ ($\gamma \in V'$) carries an H-invariant positive measure μ_γ.

(a) If Q is anisotropic, then $O(Q)$ is compact. Each π_γ has a realization inside $C^\infty(V)$.

(b) Assume Q is isotropic and $n \geqq 3$. Then:

(i) All μ_γ ($\gamma \in V'$) are tempered Radon measures.

(ii) $\mathcal{F}(L^2(\mu_\gamma)) \subset L^2_{\text{loc}}(V)$.

Here we apply some results of Schiffmann and Rallis ([6]).

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