61. *Invariants of Reductive Lie Groups of Rank One and Their Applications*

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§1. **Introduction.** Throughout this note, G will denote a reductive complex linear algebraic group. A representation of G is a finite dimensional vector space V over the complex number field $\mathbf{C}$ together with a morphism of algebraic groups $\varphi: G \to GL(V)$. We will consider φ together with the representation space V and denote a representation as above by φ, by (φ, G) or by $\varphi(G)$. Let $\mathbf{C}[\varphi]$ be the coordinate ring of the affine space φ on which G acts naturally and let $\mathbf{C}[\varphi]^G$ be the $\mathbf{C}$-subalgebra consisting of all invariant polynomials in $\mathbf{C}[\varphi]$ under this action of G. (φ, G) is said to be *completely co-intersected* (abbrev. COCI) if $\mathbf{C}[\varphi]^G$ (and so φ/G) is a complete intersection. Recall that (φ, G) is said to be *coregular* if $\mathbf{C}[\varphi]^G$ is a polynomial ring over $\mathbf{C}$. All coregular representations of simple algebraic groups were determined in [2, 10].

When Gx is closed in φ for an element x in φ, the isotropy group G_x is reductive, and we call the natural representation of G_x on $T_xV/T_x(G_x)$ *the slice representation at* x, which is denoted by φ_x. Then $\varphi_x/G_x \to \varphi/G$ is etale at the image of x in φ/G ([6]), and we easily get

Lemma (1.1). *Every slice representation of a COCI representation of G is COCI.*

As any representation of G is completely reducible, [11, (5.2)] implies

Lemma (1.2). *Every subrepresentation of a COCI representation of G is COCI.*

These lemmas are useful in studying COCI representations of reductive groups.

§2. **Reductive groups of rank one.** In this section, we suppose that $\operatorname{rank} G=1$. Let T be a maximal torus of G and $\nu: \operatorname{Hom}(T, \mathbf{C}^*) \cong \mathbf{Z}$ a fixed isomorphism. For a representation ρ of G, let ρ^+ (resp. ρ^-) be the direct sum of all ρ_χ with $\nu(\chi)>0$ (resp. $\nu(\chi)<0$), where ρ_χ is the subspace of ρ of weight $\chi \in \operatorname{Hom}(T, \mathbf{C}^*)$. Moreover, put $q_T(\rho)=\min\{\dim \rho^+, \dim \rho^-\}$ and $p_T(\rho)=\min\{|\{\chi \mid \rho_\chi \neq 0, \nu(\chi)>0\}|, |\{\chi \mid \rho_\chi \neq 0, \nu(\chi)<0\}|\}$.

Theorem (2.1). *Let ρ be a representation of G and suppose that (ρ, G) is COCI. Then:*

(1) *If* $G^0 \neq T$ *and* $q_T(\rho) \geqq 4$, *then* $q_T(\rho)=4$ *and* ρ/ρ^{G^0} *is reducible as a representation of* G^0.

(2) *If* $G^0=T$ *and* $q_T(\rho) \geqq 4$, *then* $q_T(\rho) \leqq 6$, $p_T(\rho) \leqq 1$ *and* $Z_G(T)=G$.

As a special case of (the proof) of (2.1), we have

Theorem (2.2). *Let* ρ *be a representation of* G *such that* $Z_G(T)$ *is diagonalizable on* $\rho^+ \oplus \rho^-$. *Suppose that* (ρ, G) *is COCI. Then*:

(1) *If* $q_T(\rho) \geqq 4$, *then* $G^0 \neq T$, $q_T(\rho)=4$ *and* ρ/ρ^{G^0} *is a reducible representation of* G.

(2) *If* $G=Z_G(T)$, *then* $q_T(\rho) \leqq 2$.

Remark (2.3). When we use the slice method in Sect. 3, the assumption on $Z_G(T)$ in (2.2) is not essential. In fact, finite principal closed isotropy groups of some algebraic groups are abelian.

A version (e.g. [8]) of Grothendieck's result on purity is useful in showing (2.1).

§3. Applications. For representations φ, ψ of G, we adopt the following notation: $\varphi+\psi$ (resp. $k\varphi$) denotes the direct sum of φ and ψ (resp. of k copies of φ) and $\varphi-\psi$ denotes the virtual representation in the representation ring of G satisfying $(\varphi-\psi)+\psi=\varphi$. If G is connected and φ and ψ are irreducible, $\varphi\psi$ stands for the irreducible component of highest weight in $\varphi \otimes \psi$. If H is a subgroup of G, $(\varphi(G), H)$ denotes the restriction of φ to H.

Hereafter let G be a simple, connected and simply connected algebraic group of rank r and let $\varphi_1, \cdots, \varphi_r$ denote the basic irreducible representations of G whose ordering is defined in [1]. We confuse G with its Dynkin diagram. If $\gamma: H \rightleftarrows G$, (φ, G) is identified with $\gamma(\varphi, H)$ (note $\mathbf{D}_3 \rightleftarrows \mathbf{A}_3$).

Theorem (3.1). *Let* φ *be a nontrivial irreducible representation of* G. *If* (φ, G) *is COCI and not coregular, then* (φ, G) *or its dual can be identified with one of* $\varphi_1^5(\mathbf{A}_1)$, $\varphi_1^6(\mathbf{A}_1)$, $\varphi_1^3(\mathbf{A}_3)$ *and* $\varphi_1\varphi_2(\mathbf{A}_3)$.

Corollary (3.2). *Let* φ *be a representation of* G *which does not contain a nonzero trivial subrepresentation. If* (φ, G) *is COCI and there is a non-coregular irreducible subrepresentation of* φ, *then* φ *is irreducible.*

Proof. If rank $G=1$, this is obvious (cf. (2.2)), and otherwise the assertion easily follows from (3.1) (cf. (1.1)).

Example (3.3) ([3]). $\varphi_1^i(\mathbf{A}_n)$ is COCI if and only if (i, n) equals to one of $(1, n)$, $(2, n)$, $(3, 1)$, $(4, 1)$, $(5, 1)$, $(6, 1)$, $(3, 2)$ and $(3, 3)$. This is a direct consequence of (1.1) and classical invariant theory.

The next theorem, in which it is not necessary to assume that G is simple, follows immediately from (1.1), (2.1) and [7].

Theorem (3.4). *Let* φ *be a representation of* G *and let* T' *be a maximal torus of a simple* 3-*dimensional subgroup* G' *of* G. *Suppose that*

(1) *For some* $v\in\varphi^{T'}$, $N_G(T')v$ *is closed and* $(N_G(T')/T')_v$ *is finite.*

(2) *The conclusions of* (2.1) *are not satisfied for* ρ, $T=T'$ *and* $G=G_v$. *Here* $\rho=\varphi-\mathrm{Ad}(G)$ *if* $G_v^0=T'$, *and otherwise* $\rho=\varphi-\mathrm{Ad}(G)+\mathrm{Ad}(G_v)$.

Then (φ, G) *is not COCI.*

Using (2.2), we can weaken the condition (2) in (3.4) (cf. (2.3)), but (3.4) is sufficient for showing (3.1).

Sketch of the proof of (3.1). Let φ be an irreducible representation of G which is not listed in (3.1). By (2.2) and (3.3) we may assume $r\geqq 2$ and $(\varphi, G)\neq(\varphi_1^i, \mathbf{A}_n)$. Because (φ, G) is not coregular, there is a maximal torus T' of a simple 3-dimensional subgroup G' of G satisfying (1) in (3.4) and $q_{T'}(\varphi-\mathrm{Ad}(G))\geqq 3$ (cf. [10, (5.2)]). Moreover such a torus is effectively listed in [10, §5]. Thus using [4, 5] and tensor decompositions in [10], we can estimate $q_{T'}(\varphi-\mathrm{Ad}(G))$, which implies that almost all φ's are not COCI (cf. (3.4)). (Sometimes, one needs to study the decomposition of $(\varphi(G), N_G(G')^0)$.) If necessary, we furthermore compute G_v^0 and $p_{T'}(\varphi-\mathrm{Ad}(G))$, and consequently (3.1) follows.

Example (3.5). Let $T'=\mathbf{C}^*\subset G'=\mathbf{A}_1\subset G=\mathbf{A}_n$ where $(\varphi_1(\mathbf{A}_n), \mathbf{A}_1)=k\varphi_1(\mathbf{A}_1)+1_m$, $m=k+1$, $k+2$ or $k+3$. Then $\varphi=\varphi_3(\mathbf{A}_n)$ $(n>8)$ satisfies (1) of (3.4). The inequality $q_{T'}(\varphi-\mathrm{Ad}(G))\geqq 7$ is shown as follows: For simplicity, suppose $n=9$. Because $\varphi_1^{\otimes 3}(\mathbf{A}_1)=\varphi_1^3(\mathbf{A}_1)+2\varphi_1(\mathbf{A}_1)$, $(\varphi_3(\mathbf{A}_9), G')$ contains $\varphi_1^3(G')+12\varphi_1^2(G')+26\varphi_1(G')$. Thus $q_{T'}(\varphi-\mathrm{Ad}(G))\geqq 40-q_{T'}(9\varphi_1^2(G')+24\varphi_1(G'))\geqq 7$.

Example (3.6). Let $T'=\mathbf{C}^*\subset G'=\mathbf{C}_1\subset G=\mathbf{C}_2$ and $\varphi=\varphi_2^3$ where $(\varphi_1(\mathbf{C}_2), \mathbf{C}_1)=2\varphi_1(\mathbf{C}_1)$. Then (1) of (3.4) is satisfied. Since $(\varphi(G), G')$ contains $\varphi_1^6(G')+2\varphi_1^4(G')+2\varphi_1^2(G')$ and $(\mathrm{Ad}(G), G')=3\varphi_1(G')+1$, we have $q_{T'}(\varphi-\mathrm{Ad}(G))\geqq 6$ and $p_{T'}(\varphi-\mathrm{Ad}(G))\geqq 3$, which shows that $\varphi_2^3(\mathbf{C}_2)$ is not COCI.

Example (3.7). For exceptional groups G, using the facts stated in [10, p. 189, 190], we can easily examine the conditions in (3.4). Especially if G is of type $\mathbf{E}$, our assertion is almost trivial. Let $T'=\mathbf{C}^*\subset G'=\mathbf{A}_1\subset G=\mathbf{G}_2$ where $(\varphi_1(\mathbf{G}_2), G')=2\varphi_1(G')+1_3$. Clearly $(\mathrm{Ad}(\mathbf{G}_2), G')=\varphi_1^2(G')+4\varphi_1(G')+1_m$. By the identity $(\varphi_1(\mathbf{G}_2), \mathbf{A}_1^{(1)}\times\mathbf{A}_1^{(2)})=\varphi_1(\mathbf{A}_1^{(1)})\otimes\varphi_1(\mathbf{A}_1^{(2)})+\varphi_1^2(\mathbf{A}_1^{(2)})$, $(\varphi_1^2(\mathbf{G}_2), G')\supset 3\varphi_1^2(G')+8\varphi_1(G')$. Thus $p_{T'}(\varphi_1^2-\mathrm{Ad}(\mathbf{G}_2))\geqq 2$ and $q_{T'}(\varphi_1^2-\mathrm{Ad}(\mathbf{G}_2))\geqq 6$, which implies our assertion for $\varphi=\varphi_1^2(\mathbf{G}_2)$.

Theorem (3.8). *For each coregular irreducible representation* (φ, G), *we can effectively give a number* n_φ *such that* $(n_\varphi\varphi, G)$ *is not COCI.*

Example (3.9). $(m\varphi_1, \mathbf{A}_n)$ $(n\geqq 2)$ is COCI if and only if $m\leqq n+2$.

Remark (3.10) ([9]). For a fixed G, up to outer automorphisms and additions of trivial representations, there are only finite many

COCI representations. This follows from (1.2), (3.2) and (3.8).

Since the syzygies of $\mathbf{C}[\varphi]^G$ are constructive, in principle, we can determine COCI representations of G.

Example (3.11) ([9]). Let φ be a representation of $\mathbf{A}_1$ which does not contain nonzero trivial subrepresentation. Then $(\varphi, \mathbf{A}_1)$ is COCI if and only if φ is a subrepresentation of φ_1^5, φ_1^6, $\varphi_1^2+\varphi_1^3$, $\varphi_1^2+\varphi_1^4$, $2\varphi_1+\varphi_1^2$, $\varphi_1+2\varphi_1^2$, $2\varphi_1^3$, $4\varphi_1$, $3\varphi_1^2$, $\varphi_1+\varphi_1^3$, $\varphi_1+\varphi_1^4$, $2\varphi_1^4$.

When the author was starting the study of COCI representations, he was inspired by the results in [10, 12]. The contents of this note were partly reported in [9] with some additional facts.

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