57. *The Spaces of Siegel Cusp Forms of Degree Two and the Representation of* $\mathrm{Sp}(2, \boldsymbol{F}_p)$

By Ryuji TSUSHIMA
Department of Mathematics, Gakushuin University

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Let $\mathfrak{S}_g$ be the Siegel upper half plane of degree g and let $\Gamma_g(l)$ be the principal congruence subgroup of $\mathrm{Sp}(g, \boldsymbol{Z})$ of level l. $\mathrm{Sp}(g, \boldsymbol{Z})$ acts on $\mathfrak{S}_g$ as $Z \rightarrow M \cdot Z := (AZ+B)(CZ+D)^{-1}$, for Z in $\mathfrak{S}_g$ and $M=\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in $\mathrm{Sp}(g, \boldsymbol{Z})$. Let $S_k(\Gamma_g(l))$ be the space of Siegel cusp forms of weight k with respect to $\Gamma_g(l)$. Then the modular group $\Gamma_g(1)$ acts on $S_k(\Gamma_g(l))$ as $(Mf)(M \cdot Z) = \det(CZ+D)^k f(Z)$, for M in $\Gamma_g(1)$ and f in $S_k(\Gamma_g(l))$, (usually the action of $\Gamma_g(1)$ is defined as $(f|M)(Z) := \det(CZ+D)^{-k} \cdot f(M \cdot Z)$, but we employ the above definition to use the holomorphic Lefschetz theorem, since $\Gamma_g(1)$ acts on $\mathfrak{S}_g$ on the left. It holds that $Mf = f|M^{-1}$).

Since $\Gamma_g(l)$ acts trivially, we get a representation of the factor group $\Gamma_g(1)/\Gamma_g(l) \simeq \mathrm{Sp}(g, \boldsymbol{Z}/l\boldsymbol{Z})$. In this note we discuss the decomposition of this representation into irreducible representations. E. Hecke settled this problem in the case where $g=1$ and $l=$ a prime p $(\geqq 5)$. There are two irreducible representations of $SL(2, \boldsymbol{F}_p)$: R_1 and R_2 of rank $(p+\varepsilon)/2$ which are dual to each other, where ε is the Legendre symbol $(-1/p)$. E. Hecke proved that if $m(R_i)$ $(i=1,2)$ is the multiplicity of R_i, then the difference $m(R_1)-m(R_2)$ is equal to 0, if $p \equiv 1 \pmod 4$ and equal to the class number $h(-p)$ of $\boldsymbol{Q}(\sqrt{-p})$, if $p \equiv 3 \pmod 4$ ([2], [3]). H. Yoshida and H. Saito obtained similar results in the Hilbert modular case ([9], [5]).

In the case of degree two, the author computed the trace of the action on $S_k(\Gamma_2(l))$ for every element of $\Gamma_2(1)/\Gamma_2(l)$ by using the holomorphic Lefschetz theorem ([7]). So if l is a prime p $(\geqq 5)$, then we can compute the multiplicity of an irreducible representation of $\mathrm{Sp}(2, \boldsymbol{F}_p)$ by the character table of B. Srinivasan ([6]) (although it is too complicated!). In the following we use the same symbols for the conjugacy classes and irreducible representations of $\mathrm{Sp}(2, \boldsymbol{F}_p)$ as in [6]. Before this work, T. Yamazaki computed the trace of the action of the conjugacy classes D_{31}, D_{32}, D_{33} and D_{34} in the case when l is a prime p and from this he computed $m(\theta_9)+m(\theta_{10})-m(\theta_{11})-m(\theta_{12})$, where $m(\ )$ means the multiplicity ([8]). But this result was false. In [4] R. Lee and S. H. Weintraub computed differences of the traces of some elements of $\Gamma_2(1)/\Gamma_2(p)$ and proved that they are equal to products of polynomials of p and $h(-p)$. But it is rather strange that they did not refer to my paper [7] although the first named author of [4] had gotten the preprint of [7] sixteen months before the date of the communication of [4]. In this note we correct T. Yamazaki's result by using my result [7] and present a conjecture that the above value is represented by the square of $h(-p)$.

T. Yamazaki observed the following lemma from [6]:

Lemma 1. *For a prime $p\geqq 3$, it holds that*

$$m(\theta_9)+m(\theta_{10})-m(\theta_{11})-m(\theta_{12})$$
$$=(1/2p)(\mathrm{trace}\, D_{31}+\mathrm{trace}\, D_{34}-\mathrm{trace}\, D_{32}-\mathrm{trace}\, D_{33}).$$

In [7] all elements of $\mathrm{Sp}\,(2, \boldsymbol{F}_p)/(\pm 1)$ which have fixed points in the smooth compactification of the quotient space of $\mathfrak{S}_g$ by $\Gamma_2(l)$ were classified and indexed as $\varphi_1, \varphi_2, \cdots, \varphi_{25}$. In $\mathrm{Sp}\,(2, \boldsymbol{F}_p)/(\pm 1)$, the conjugacy classes $\pm D_{32}$ and $\pm D_{33}$ coincide with each other. The following lemma is easily proved:

Lemma 2. *Assume that $p\geqq 5$. Then the set of φ_i's $(i=1, 2, \cdots, 25)$ which are conjugate to $\pm D_{31}$, $\pm D_{32}$ or $\pm D_{34}$ is as follows.*

1) *$\varphi_{22}(3, r, t)$ is conjugate to $\pm D_{31}$ or to $\pm D_{34}$.*

2) *$\varphi_{23}(4, r, t)$ is conjugate to $\pm D_{31}$ or to $\pm D_{34}$, if $(rt/p)=1$ and conjugate to $\pm D_{32}$, if $(rt/p)=-1$.*

3) *$\varphi_{24}(4, r, t)$ is conjugate to $\pm D_{31}$ or to $\pm D_{34}$, if $(rt/p)=1$ and conjugate to $\pm D_{32}$, if $(rt/p)=-1$.*

4) *$\varphi_{25}(4, r, s, t)$ is conjugate to $\pm D_{31}$ or to $\pm D_{34}$, if $\left(\frac{(r+t+2s)(r+t-2s)}{p}\right)=1$ and conjugate to $\pm D_{32}$, if $\left(\frac{(r+t+2s)(r+t-2s)}{p}\right)=-1$ and not conjugate to $\pm D_{31}$, $\pm D_{32}$ or $\pm D_{34}$, if $r+t-2s$ is congruent to 0 modulo p.*

Let ζ be $\exp(2\pi i/p)$. From these two lemmas and [7] Theorem (5.1) we derive the following

Theorem. *If $p\geqq 5$ and $k\geqq 4$, then*

$$m(\theta_9)+m(\theta_{10})-m(\theta_{11})-m(\theta_{12})$$
$$=\frac{(-1)^k}{2p}\left\{\sum_{r=1}^{p-1}\left(\frac{4}{(\zeta^r-1)^2}+\frac{3}{(\zeta^r-1)}\right)+\sum_{r=1,\, t=1}^{p-1,\, p-1}\left(\frac{rt}{p}\right)\frac{1}{(\zeta^r-1)(\zeta^t-1)}\right.$$
$$\left.+2\sum_{r=0,\, t=1,\, (r+2t)(r-2t)\not\equiv 0}^{p-1,\, p-1}\left(\frac{(r+2t)(r-2t)}{p}\right)\frac{1}{(\zeta^{r+2t}-1)(\zeta^{-t}-1)}\right\}.$$

In the theorem the first term is the contribution of φ_{22}, the second term is the contribution of φ_{23} and φ_{24}, and the third term is the contribution of φ_{25}. T. Yamazaki mistook the sign of the first and the third terms. We can easily express this in a form not including ζ. For an integer n, we denote by $\langle n\rangle$ the integer such that $0\leqq\langle n\rangle\leqq p-1$

and $n\equiv\langle n\rangle \pmod p$, and we denote by 4^{-1} an integer which is the inverse of 4 modulo p. Then the above formula is transformed to

$$\frac{(-1)^k}{2p}\left\{\frac{(p-1)(1-2p)}{6}-\frac{1}{p}\left(\sum_{k=1}^{p-1}\left(\frac{k}{p}\right)k\right)^2\right.$$
$$\left.+\frac{2}{p}\sum_{k=1,\,a=1}^{p-1,\,p-1}\left(\frac{k}{p}\right)a\langle -a(k-4^{-1})\rangle+\frac{(p-1)^2}{2}\right\}.$$

The first, the second and the third terms correspond to the first, the second and the third terms in the theorem, respectively. Now we present the following

Conjecture. *If $p\geqq 5$ and $k\geqq 4$, then*

$$m(\theta_9)+m(\theta_{10})-m(\theta_{11})-m(\theta_{12})=\begin{cases}(-1)^{k+1}(h(-p))^2, & \text{if } p\equiv 7 \pmod 8\\ (-1)^{k+1}(2h(-p))^2, & \text{if } p\equiv 3 \pmod 8\\ (-1)^{k+1}(h(-p)/2)^2, & \text{if } p\equiv 1 \pmod 4.\end{cases}$$

The author verified this conjecture for prime numbers smaller than 500 by using computer.

Remark. The author obtained the above conjecture in March 1983 but he could not prove it. This conjecture was proved by K. Hashimoto in [1] by a quite different method which uses the Selberg's trace formula after the author wrote the manuscript of this note and sent it to him in October 1983. But the author does not know its direct proof from the above expression. In the method of Selberg's trace formula, the traces of the action of the elements of $\mathrm{Sp}(2, \boldsymbol{F}_p)$ are represented by special values of certain L-functions. But it is very difficult to evaluate these values in general. So it would be interesting to compare them with the results in [7].

References

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