41. Local Existence of C^∞-Solution for the Initial-Boundary Value Problem of Fully Nonlinear Wave Equation

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We shall consider the local existence in time of C^∞-solutions for the following initial-boundary value problem:

$$\text{(M.P)} \qquad \begin{aligned} &\mathcal{L}u+F(t,x,\bar{D}^2u)=f(t,x) \quad \text{in } [0,T]\times\Omega,\\ &u=0 \quad \text{on } [0,T]\times\partial\Omega,\\ &u(0,x)=\psi_0(x), \quad (\partial_t u)(0,x)=\psi_1(x) \quad \text{in } \Omega, \end{aligned}$$

where

$$\mathcal{L}v=\partial_t^2 v+a_1(t,x,\bar{D}_x^1)\partial_t v+a_2(t,x,\bar{D}_x^2)v,$$
$$a_1(t,x,\bar{D}_x^1)v=\textstyle\sum_{j=1}^n a_2^j(t,x)\partial_j v+a_1^0(t,x)v,$$
$$a_2(t,x,\bar{D}_x^2)v=-\textstyle\sum_{i,j=1}^n a_2^{ij}(t,x)\partial_i\partial_j v+\sum_{j=1}^n a_1^j(t,x)\partial_j v+a_0(t,x)v,$$

and $a_2(t,x,\bar{D}_x^2)$ is a strictly elliptic operator with $a_2^{ij}=a_2^{ji}$. Here and hereafter we use the notations:

$$\partial_t=\partial_0=\partial/\partial t, \quad \partial_j=\partial/\partial x_j, \quad \partial_x^\alpha=\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n} \quad (|\alpha|=\alpha_1+\cdots+\alpha_n),$$

and for any integer $L\geqq 0$

$$D^L v=(\partial_t^j\partial_x^\alpha v\,;\, j+|\alpha|=L), \qquad \bar{D}^L v=(\partial_t^j\partial_x^\alpha v\,;\, j+|\alpha|\leqq L),$$
$$D_x^L v=(\partial_x^\alpha v\,;\,|\alpha|=L), \qquad \bar{D}_x^L v=(\partial_x^\alpha v\,;\,|\alpha|\leqq L).$$

Ω is a domain in $\boldsymbol{R}^n$ with compact and C^∞ boundary $\partial\Omega$. Let T be some positive constant.

In the case of $\Omega=\boldsymbol{R}^n$ the local existence in time of C^∞-solutions of fully nonlinear wave equations is already known (see, e.g., [2]), since we can reduce fully nonlinear equations to quasilinear systems by the method due to Dionne [1], the local solvability of which has extensively been studied (see, e.g., Kato [3] and [4]). However, we can not apply that method to the initial-boundary value problem. Accordingly, for the initial-boundary value problem the Nash-Moser technique has often been used in order to overcome the so-called derivative loss which results from the fully nonlinearity of the equation (see, e.g., [5], [7], [9], [10] and [11]). Moreover, because of the difficulty of the derivative loss it has been unknown whether the C^∞-solution exists or not even when $\psi_0(x)$, $\psi_1(x)$ and $f(t,x)$ are in a class of C^∞.

In the present paper we give the local existence theorem of C^∞-solutions of Problem (M.P). Our method is essentially based on the

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ellipticity of the differential operator $a_2(t, x, \bar{D}_x^2)$ in the equation (M.P), and the Nash-Moser technique is not used. We make the equation (M.P) a coupled system of a nonlinear wave equation and a nonlinear elliptic equation to overcome the difficulty of the derivative loss. The details of the proof will appear elsewhere.

We first list notations. For p with $1 \leqq p \leqq \infty$ $L^p(\Omega)$ and $\|\cdot\|_p$ denote the usual L^p function space defined on Ω and its norm, respectively. For a vector-valued function $f=(f_1, \cdots, f_s)$ we put $\|f\|_p=\|f_1\|_p+\cdots+\|f_s\|_p$. For a nonnegative integer L we put

$$H^L(\Omega)=\{v \in L^2(\Omega)\,;\, \|\bar{D}_x^L v\|_2<+\infty\} \quad \text{and} \quad \|v\|_{2,L}=\|\bar{D}_x^L v\|_2.$$

Especially $H^\infty(\Omega)$ denotes $\bigcap_{L=1}^\infty H^L(\Omega)$. We denote the completion in $H^1(\Omega)$ of $C_0^\infty(\Omega)$ by $H_0^1(\Omega)$. For a nonnegative integer L, $\mathfrak{B}^L(\bar{\mathcal{O}})$ denotes the set of all functions having all derivatives of order $\leqq L$ continuous and bounded in $\bar{\mathcal{O}}$, where $\mathcal{O}$ is Ω or $(0, T)\times\Omega$. For $-\infty \leqq a < b \leqq +\infty$, a nonnegative integer k and a Banach space E, $C^k([a, b]\,;\, E)$ denotes the set of all E-valued functions having all derivatives of order $\leqq k$ continuous in $[a, b]$. For $u \in \bigcap_{j=0}^L C^j([a, b]\,;\, H^{L-j}(\Omega))$ we put

$$|u|_{L,[a,b]}=\sup_{a\leqq t\leqq b}\|\bar{D}^L u(t)\|_2.$$

For positive integers s, i, a function $H=H(t, x, \nu)$, $\nu=(\nu_1, \cdots, \nu_s)$, defined on $[0, T]\times\bar{\Omega}\times \boldsymbol{R}^s$, vectors $u=(u_1, \cdots, u_s)$, $v_j=(v_1^j, \cdots, v_s^j) \in \boldsymbol{R}^s$, we put

$$(d_\nu^i H)(t, x, u)(v_1, \cdots, v_i)=(\partial^i H/\partial\eta_1\cdots\partial\eta_i)(t, x, u+\textstyle\sum_{j=1}^i \eta_j v_j)|_{\eta_1=\cdots=\eta_i=0}.$$

We next make the following assumptions on $\mathcal{L}$ and F.

Assumption [A]. (1) The coefficients $a_2^j(t, x)$, $a_2^{ij}(t, x)$, $a_1^j(t, x)$ and $a_0(t, x)$ of $\mathcal{L}$ are real-valued functions belonging to $\mathfrak{B}^\infty([0, T]\times\bar{\Omega})$.

(2) $F(t, x, \lambda)$ is a real-valued function defined on

$$[0, T]\times\bar{\Omega}\times\{\lambda \in \boldsymbol{R}^{(n+1)(n+2)+1}\,;\, |\lambda|\leqq 3\lambda_0\}$$

for some $\lambda_0>0$ such that all its derivatives of any order and itself are continuous and bounded, $F(t, x, 0)=0$ and $(d_\lambda F)(t, x, 0)=0$.

(3) Functions F_2^j, F_2^{ij}, F_1^j and F_0 are defined as follows:

$$\begin{aligned}(d_\lambda F)(t, x, \bar{D}^2 u)\bar{D}^2 v=&\textstyle\sum_{j=0}^n F_2^j(t, x, \bar{D}^2 u)\partial_j\partial_t v\\ &-\textstyle\sum_{i,j=1}^n F_2^{ij}(t, x, \bar{D}^2 u)\partial_i\partial_j v\\ &+\textstyle\sum_{j=0}^n F_1^j(t, x, \bar{D}^2 u)\partial_i v+F_0(t, x, \bar{D}^2 u)v.\end{aligned}$$

Then $F_2^{ij}(t, x, \lambda)=F_2^{ji}(t, x, \lambda)$ and there exists a positive constant d such that

$$\textstyle\sum_{i,j=1}^n [a_2^{ij}(t, x)+F_2^{ij}(t, x, \lambda)]\xi_i\xi_j \geqq 2d|\xi|^2, \qquad 1+F_2^0(t, x, \lambda)\geqq 2d$$

for all $(t, x) \in [0, T]\times\bar{\Omega}$, $|\lambda|\leqq 3\lambda_0$ and $\xi \in \boldsymbol{R}^n$.

Before we state the theorem, we define a certain class of data as follows.

Definition. We shall say that a pair of functions $(\psi_0(x), \psi_1(x), f(t, x))$ with $\psi_0(x) \in \mathfrak{B}^2(\bar{\Omega})$, $\psi_1(x) \in \mathfrak{B}^1(\bar{\Omega})$ and $f(0, x) \in \mathfrak{B}^0(\bar{\Omega})$ belongs to $\mathcal{D}$ if there exists a $\psi_2(x) \in \mathfrak{B}^0(\bar{\Omega})$ such that

$$\|\bar{D}_x^2\psi_0\|_\infty+\|\bar{D}_x^1\psi_1\|_\infty+\|\psi_2\|_\infty\leqq\lambda_0$$

and

$$\psi_2(x)+a_1(0,x,\bar{D}_x^1)\psi_1(x)+a_2(0,x,\bar{D}_x^2)\psi_0(x)$$
$$+F(0,x,\bar{D}_x^2\psi_0(x),\bar{D}_x^1\psi_1(x),\psi_2(x))=f(0,x) \qquad \text{in } \Omega.$$

Now we state the following local existence theorem.

Theorem. (a) *We assume that Assumption* [A] *holds. Let* Ω *be a domain in* $\boldsymbol{R}^n$ *with compact and* C^∞ *boundary* $\partial\Omega$. *We put* $L_0=\max(2[n/2]+4,[n/2]+7)$. *Let* L *be any integer with* $L\geqq L_0$. *Let* $\psi_0\in H^{2L+2}(\Omega)$, $\psi_1\in H^{2L+1}(\Omega)$ *and*

$$f\in C^{2L+1}([0,T];L^2(\Omega))\cap\{\cap_{j=0}^{2L}C^j([0,T];H^{2L-j}(\Omega))\}$$

and let $(\psi_0,\psi_1,f)\in\mathcal{D}$ *satisfy the compatibility condition of order* $2L+1$. *Then there exists a* T_0 *with* $0<T_0\leqq T$ *depending only on* n, Ω, $\|\psi_0\|_{2,2L_0+2}$, $\|\psi_1\|_{2,2L_0+1}$, $|f|_{2L_0,[0,T]}$, F *and* $\mathcal{L}$ *such that Problem* (M.P) *has a unique local solution* $u(t,x)$:

$$u(t,\cdot)\in\{\cap_{j=0}^{2L+1}C^j([0,T_0];H^{2L+2-j}(\Omega)\cap H_0^1(\Omega))\}\cap C^{2L+2}([0,T_0];L^2(\Omega)).$$

(b) *In addition to the assumptions of* (a), *let* $\psi_0\in H^\infty(\Omega)$, $\psi_1\in H^\infty(\Omega)$ *and* $f\in\cap_{j=1}^\infty C^\infty([0,T];H^j(\Omega))$, *and let* ψ_0, ψ_1 *and* f *satisfy the compatibility condition of order infinity. Then the above local solution is in* $C^\infty([0,T_0]\times\bar{\Omega})$.

Remark. (1) In the statement of the above theorem the compatibility condition of order $2L+1$ means that the boundary values of $\partial_t^j u|_{t=0}$ $(0\leqq j\leqq 2L+1)$ are compatible with the boundary condition. For details of the compatibility condition, see [9, §2.3], [10, §10] and [11, §4.2].

(2) We note that there actually exist the non-zero data $(\psi_0(x),\psi_1(x),f(t,x))\in\mathcal{D}$ satisfying the compatibility condition of order infinity under Assumption [A]. For example, it is satisfied if $\psi_0(x)\in C_0^\infty(\Omega)$, $\psi_1(x)\in C_0^\infty(\Omega)$ and $f(0,x)\in C_0^\infty(\Omega)$ are sufficiently small in certain norms (see [9, §2.3] and [10, p. 44]).

(3) We note that T_0 is independent of the size of $\|\psi_0\|_{2,2N+2}$, $\|\psi_1\|_{2,2N+1}$ and $|f|_{2N,[0,T]}$ for an integer $N>L_0$.

(4) By combining the above theorem and the results of [11] we obtain a unique global C^∞-solution for Problem (M.P) with $\mathcal{L}=\partial_t^2-\Delta$, if the data are sufficiently smooth and small and the domain Ω satisfies certain conditions (for details, see [11]).

(5) Our method is essentially based on the ellipticity of the differential operator $a_2(t,x,\bar{D}_x^2)$ in $\mathcal{L}$. We also have results analogous to the above theorem for the nonlinear Klein-Gordon equation and the nonlinear Schrödinger equation.

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