40. On the Euler-Poisson-Darboux Equation and the Toda Equation. I

By Yoshinori KAMETAKA
Department of Applied Mathematics, Faculty of Engineering, Ehime University

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§1. **Summary.** The Toda equation with two time variables

(1.1) $$XY \log t_n = t_{n+1}t_{n-1}/t_n^2 \qquad \left(X=\frac{\partial}{\partial x},\ Y=\frac{\partial}{\partial y},\ t_n=t_n(x,y)\right)$$

can be solved using solutions of the Euler-Poisson-Darboux equations ([1])

(1.2) $$(XY+(\alpha+\beta-1-2n)\varphi^{-1}X-(n-\alpha)(n-\beta)\varphi^{-2})u_n=0,$$

where $\varphi=x-y$. Rational solutions, Gauss hypergeometric function solutions and solutions which can be expressed by hypergeometric functions with two variables (Appell hypergeometric functions F_1, F_2 and F_3 are included) are obtained. K. Okamoto ([2]) also found these hypergeometric solutions.

§2. **Bäcklund transformation.** When t_n satisfies (1.1)

(2.1) $$r_n=XY\log t_n, \qquad s_n=Y\log t_{n-1}/t_n$$

satisfies

(2.2) $$Yr_n=r_n(s_n-s_{n+1}), \qquad Xs_n=r_{n-1}-r_n.$$

Eliminating s_n we have

(2.3) $$XY\log r_n=r_{n+1}-2r_n+r_{n-1}.$$

This form of the Toda equation was found by G. Darboux ([1]). As was shown in our previous work ([3])

(2.4) $$t_n=F(n)\varphi^{-f(n)},$$

where $f(n)=(n-\alpha)(n-\beta)$, α and β are arbitrary constants,

$$F(n+1)F(n-1)/F(n)^2=-f(n), \qquad F(0)=F(1)=1,$$

satisfies the Toda equation (1.1). Corresponding

(2.5) $$r_n=-f(n)\varphi^{-2}, \qquad s_n=(\alpha+\beta+1-2n)\varphi^{-1}$$

satisfies the Toda equation (2.2). This simple important solution r_n was first found by G. Darboux ([1]). For these special solutions we put

(2.6) $$M_n=XY+(\alpha+\beta-1-2n)\varphi^{-1}X-(n-\alpha)(n-\beta)\varphi^{-2},$$
$$X_n=((n-\alpha)(n-\beta))^{-1}\varphi^2X, \qquad Y_n=Y+(\alpha+\beta-1-2n)\varphi^{-1}.$$

Define

(2.7) $$T=\{u_n\,;\,M_0u_0=0,\ u_{n+1}=Y_nu_n\ (n\geqslant 0),\ u_{n-1}=X_nu_n\ (n\leqslant 0)\}.$$

Theorem 2.1 (Bäcklund transformation). *If $u_n\in T$ then we have $M_nu_n=0$, $u_{n+1}=Y_nu_n$, $u_{n-1}=X_nu_n$ $(n=0,\pm 1,\pm 2,\cdots)$ and $\tau_n=u_nt_n$*

satisfies the Toda equation (1.1).

§3. One-parameter groups on T. We can determine all of the first order partial differential operators $D=a(x,y)X+b(x,y)Y+c(x,y)$ which commute with M_0 (modulo M_0).

Theorem 3.1.

(3.1) $A=X+Y,\quad B=x^2X+y^2Y+(1-\alpha-\beta)y,\quad C=xX+yY$

commute with M_0 (modulo M_0). More precisely

(3.2) $[M_0,A]=0,\quad [M_0,B]=2(x+y)M_0,\quad [M_0,C]=2M_0,$

where $[M,D]=MD-DM$. We have the following expression.

(3.3) $\varphi^2M_0=BA-(C-\alpha)(C-\beta)=AB-(C+1-\alpha)(C+1-\beta).$

A, B and C keep $\ker M_0$ *invariant.*

We can construct three one-parameter groups of linear transformations and a finite group which keep T invariant.

Theorem 3.2 (Main theorem). *If $u_n\in T$ then*

(3.4) $\tilde{A}(\lambda)u_n(x,y)=u_n(x+\lambda,\ y+\lambda),$

$\tilde{B}_n(\mu)u_n(x,y)=(1-\mu y)^{\alpha+\beta-1-2n}u_n(x/(1-\mu x),\ y/(1-\mu y)),$

$\tilde{C}_n(\nu)u_n(x,y)=e^{n\nu}u_n(e^\nu x,\ e^\nu y),$

(3.5) $R_nu_n(x,y)=(-y)^{\alpha-n}y^{\beta-1-n}u_n(x^{-1},y^{-1})$

belong to T. $\tilde{A}(\lambda)$, $\tilde{B}_n(\mu)$ and $\tilde{C}_n(\nu)$ are one-parameter groups of linear transformations with generators

(3.6) $A=X+Y,\quad B_n=x^2X+y^2Y+(2n+1-\alpha-\beta)y,\quad C_n=xX+yY+n,$

respectively. Each of these one-parameter groups and their corresponding generators keep $\ker M_n$ *invariant.* $\{R_n^2=\mathrm{id.}, R_n\}$ *is a finite group.*

We can show the following commutation relations.

Theorem 3.3 (Commutation relations). *For any values of complex numbers λ, μ and ν we have*

(3.7) $\tilde{A}(\lambda)\tilde{C}_n(\nu)=\tilde{C}_n(\nu)\tilde{A}(e^\nu\lambda),\qquad \tilde{B}_n(\mu)\tilde{C}_n(\nu)=\tilde{C}_n(\nu)\tilde{B}_n(e^{-\nu}\mu),$

$\tilde{A}(\lambda)\tilde{B}_n(\mu)=(1-\lambda\mu)^{\alpha+\beta-1}\tilde{B}_n(\mu/(1-\lambda\mu))$

$\times\tilde{C}_n(-2\log(1-\lambda\mu))\tilde{A}(\lambda/(1-\lambda\mu)),$

(3.8) $A\tilde{C}_n(\nu)=e^\nu\tilde{C}_n(\nu)A,\qquad \tilde{A}(\lambda)C_n=(C_n+\lambda A)\tilde{A}(\lambda),$

$B_n\tilde{C}_n(\nu)=e^{-\nu}\tilde{C}_n(\nu)B_n,\qquad \tilde{B}_n(\mu)C_n=(C_n-\mu B_n)\tilde{B}_n(\mu),$

$A\tilde{B}_n(\mu)=\tilde{B}_n(\mu)\{A+\mu(2C_n+1-\alpha-\beta)+\mu^2B_n\},$

$\tilde{A}(\lambda)B_n=\{B_n+\lambda(2C_n+1-\alpha-\beta)+\lambda^2A\}\tilde{A}(\lambda),$

(3.9) $AC_n=(C_n+1)A,\qquad B_nC_n=(C_n-1)B_n,$

$AB_n=B_nA+2C_n+1-\alpha-\beta,$

(3.10) $R_n\tilde{B}_n(\mu)=\tilde{A}(-\mu)R_n,\qquad R_n\tilde{C}_n(\nu)=e^{(\alpha+\beta-1)\nu}\tilde{C}_n(-\nu)R_n,$

(3.11) $R_nB_n=-AR_n,\qquad R_nC_n=-(C_n+1-\alpha-\beta)R_n.$

§4. Eigenfunctions. Eigenfunctions of C_n are given by Gauss hypergeometric functions $F(\alpha,\beta,\gamma;z)$. Abbreviation $(a)_n=\Gamma(n+a)/\Gamma(a)$ is used.

Theorem 4.1. *Dimension of the vector space $T\cap\{u_n\in\ker(C_n-\gamma)\}$ is 2. Its bases are given by*

$$(4.1)\quad f_n(\alpha,\beta,\gamma\,;x,y) =(\gamma+1-\alpha-\beta)_n(y-x)^{\beta-n}y^{\gamma-\beta}F(\beta-\gamma,\beta-n,\alpha+\beta-\gamma-n\,;x/y),$$

$$g_n(\alpha,\beta,\gamma\,;x,y)=\frac{(1-\alpha)_n(1-\beta)_n}{(\gamma+2-\alpha-\beta)_n}(y-x)^{\beta-n}x^{n+1+\gamma-\alpha-\beta}y^{\alpha-1-n} \times F(n+1-\alpha,\gamma+1-\alpha,n+2+\gamma-\alpha-\beta\,;x/y).$$

We have the following relations.

$$(4.2)\quad (-A)^k f_n(\alpha,\beta,\gamma\,;x,y)=\frac{(\alpha-\gamma)_k(\beta-\gamma)_k}{(\alpha+\beta-\gamma)_k}f_n(\alpha,\beta,\gamma-k\,;x,y),$$

$$B_n^k f_n(\alpha,\beta,\gamma\,;x,y)=(\gamma+1-\alpha-\beta)_k f_n(\alpha,\beta,\gamma+k\,;x,y),$$

$$(-A)^k g_n(\alpha,\beta,\gamma\,;x,y)=(\alpha+\beta-\gamma-1)_k g_n(\alpha,\beta,\gamma-k\,;x,y),$$

$$B_n^k g_n(\alpha,\beta,\gamma\,;x,y)=\frac{(\gamma+1-\alpha)_k(\gamma+1-\beta)_k}{(\gamma+2-\alpha-\beta)_k}g_n(\alpha,\beta,\gamma+k\,;x,y).$$

Eigenfunctions of A and B_n are given by confluent hypergeometric functions $F(\alpha,\beta\,;z)$.

Theorem 4.2. *Put*

$$(4.3)\quad h_n(\alpha,\beta\,;x,y)=(1-\beta)_n(y-x)^{\alpha-n}e^yF(\alpha-n,1+\alpha-\beta\,;x-y).$$

$T\cap\{u_n\in\ker(A-1)\}$ *is a 2-dimensional vector space. Its bases are given by* $h_n(\alpha,\beta\,;x,y)$ *and* $h_n(\beta,\alpha\,;x,y)$. $\tilde{C}_n(\nu)h_n(\alpha,\beta\,;x,y)$ *and* $\tilde{C}_n(\nu)h_n(\beta,\alpha\,;x,y)$ *are bases of the 2-dimensional vector space* $T\cap\{u_n\in\ker(A-e^\nu)\}$. $R_nh_n(\alpha,\beta\,;x,y)$ *and* $R_nh_n(\beta,\alpha\,;x,y)$ *are bases of the 2-dimensional vector space* $T\cap\{u_n\in\ker(B_n+1)\}$. $\tilde{C}_n(\nu)R_nh_n(\alpha,\beta\,;x,y)$ *and* $\tilde{C}_n(\nu)R_nh_n(\beta,\alpha\,;x,y)$ *are bases of the 2-dimensional vector space* $T\cap\{u_n\in\ker(B_n+e^{-\nu})\}$.

§5. Rational solutions. Put

$$(5.1)\quad p_n=f_n(\alpha,\beta,\alpha\,;x,y)=(1-\beta)_n(y-x)^{\alpha-n},$$

$$q_n=R_np_n=(1-\beta)_n(y-x)^{\alpha-n}x^{n-\alpha}y^{\beta-1-n}.$$

Theorem 5.1 (Rational solutions). *For* $k=0,1,2,\cdots$

$$(5.2)\quad P_{n,k}=B_n^kp_n/p_n=(n+1-\beta)_ky^kF(-k,\alpha-n,\beta-n-k\,;x/y)$$

is a homogeneous polynomial in (x,y) *of order* k.

$$(5.3)\quad \rho_n=XY\log P_{n,k}-(n-\alpha)(n+1-\beta)\varphi^{-2} =-(n-\alpha)(n+1-\beta)\varphi^{-2}P_{n+1,k}P_{n-1,k}/P_{n,k}^2,$$

$$\sigma_n=Y\log P_{n-1,k}/P_{n,k}+(\alpha+\beta-2n)\varphi^{-1}$$

is a rational solution of the Toda equation (2.2).

$$\tilde{P}_{n,k}=C_n^k\tilde{B}_n(\mu)p_n/\tilde{B}_n(\mu)p_n,\qquad Q_{n,k}=A^kq_n/q_n$$

and

$$\tilde{Q}_{n,k}=C_n^k\tilde{A}(\lambda)q_n/\tilde{A}(\lambda)q_n$$

are also essentially polynomials and give rational solutions of the Toda equation.

§6. Hypergeometric solutions. By eigenfunction expansion we can construct various solutions of the Toda equation. If

$$(6.1)\quad u_n=\textstyle\sum_{k=0}^\infty a_kf_n(\alpha,\beta,\gamma-\varepsilon k\,;x,y)\qquad (\varepsilon \text{ is an integer})$$

converges then it belongs to T. If we choose ε and a_k suitably then we can express u_n by hypergeometric functions with two variables of order two which we can find in Horn's list ([4]).

Theorem 6.1 (Hypergeometric solutions). *When* $\varepsilon=1$ *and* $a_k=(\beta')_k(\beta-\gamma)_k/(\alpha+\beta-\gamma)_k k!$ *we have*

$$(6.2)\qquad \begin{aligned} u_n &= (\gamma+1-\alpha-\beta)_n(y-x)^{\beta-n}y^{\gamma-\beta} \\ &\quad \times F_1(\beta-\gamma, \beta-n, \beta', \alpha+\beta-\gamma-n; x/y, 1/y) \\ &= F(\beta', \alpha-\gamma; -A)f_n(\alpha, \beta, \gamma; x, y), \end{aligned}$$

where

$$F_1(\alpha, \beta, \beta', \gamma; x, y)=\sum_{j,k}\frac{(\alpha)_{j+k}(\beta)_j(\beta')_k}{(\gamma)_{j+k}j!\,k!}x^j y^k$$

is one of the Appell's hypergeometric functions.

Further list of hypergeometric solutions will be seen in the next paper [5].

References

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