39. *Structure of the Solution Space of Witten's Gauge-field Equations*

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§0. Introduction. Consider a gauge field in the eight-dimensional space satisfying

$$(1)\qquad \begin{aligned}&[\nabla_{y_\mu}, \nabla_{y_\nu}]=(1/2)\varepsilon_{\mu\nu\alpha\beta}[\nabla_{y_\alpha}, \nabla_{y_\beta}],\\ &[\nabla_{z_\mu}, \nabla_{z_\nu}]=(-1/2)\varepsilon_{\mu\nu\alpha\beta}[\nabla_{z_\alpha}, \nabla_{z_\beta}],\\ &[\nabla_{y_\mu}, \nabla_{z_\nu}]=0, \qquad (\mu, \nu=0,1,2,3)\end{aligned}$$

where $(y, z)=(y_0, y_1, y_2, y_3, z_0, z_1, z_2, z_3)\in C^8$, Δ_{y_μ} etc. are covariant derivatives, and $\varepsilon_{\mu\nu\alpha\beta}$ is the totally anti-symmetric tensor such that $\varepsilon_{0,1,2,3}=1$.

Set $x=(y+z)/2$, $w=(y-z)/2$. E. Witten [1] pointed out that (1) implies the second-order Yang-Mills equations

$$(2)\qquad [\nabla_{x_\mu}, [\nabla_{x_\mu}, \nabla_{x_\nu}]]=0 \qquad (\nu=0,1,2,3)$$

on the diagonal subspace $\Delta=\{(y, z)\,|\, w=0\}$, and further, that a gauge field on Δ satisfies (2) if and only if it can be extended to a neighborhood of Δ consistently to (1) mod $(w_0, w_1, w_2, w_3)^2$. Here $(w_0, w_1, w_2, w_3)^2$ denotes the square of the ideal generated by w_0, w_1, w_2, w_3.

In this paper, we rewrite (1) in the language of Sato's soliton theory [2] and investigate the structure of the solution space of (1) on the analogy of Takasaki's work ([3], [4]): we solve an initial-value problem of differential equations with respect to functions with value in an infinite-dimensional Grassmann manifold. (See Theorem 2.)

In our case, there appear a pair of spectral parameters λ_1, λ_2. The main difference from the case of one spectral parameter is that the initial data must satisfy a system of differential equations if the problem is solvable. (See Proposition 4 and cf. [3].)

§1. Linearization. Set $\eta_1=y_0+iy_1$, $\zeta_1=y_2-iy_3$, $\eta_2=z_0+iz_1$, $\zeta_2=z_2-iz_3$, $\bar\eta_1=y_0-iy_1$, $\bar\zeta_1=y_2+iy_3$, $\bar\eta_2=z_0-iz_1$, and $\bar\zeta_2=z_2+iz_3$. Then, introducing parameters λ_1, λ_2, we can rewrite (1) as follows: for any $\lambda_1, \lambda_2\in C$,

$$(3)\qquad \begin{aligned}&[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1}, \lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1}]=0, \qquad [-\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}, \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1}, -\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}]=[-\lambda_1\nabla_{\eta_1}+\nabla_{\bar\zeta_1}, \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,\\ &[\lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1}, -\lambda_2\nabla_{\eta_2}+\nabla_{\zeta_2}]=[\lambda_1\nabla_{\zeta_1}+\nabla_{\bar\eta_1}, \lambda_2\nabla_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0.\end{aligned}$$

Throughout this paper we discuss in the category of formal power series, so that $\nabla_{\eta_1}=\partial_{\eta_1}+A_{\eta_1}$, $A_{\eta_1}\in \mathrm{gl}\,(n, C[[\eta_1, \zeta_1, \cdots, \bar\zeta_2]])$ etc.

Now we "fix" the gauge, namely, restrict the freedom of gauge so that $A_{\eta_1}=A_{\zeta_1}=A_{\eta_2}=A_{\bar\zeta_2}=0$. Then (3) reads

(3′) $[-\lambda_1\partial_{\eta_1}+\nabla_{\xi_1},\ \lambda_1\partial_{\zeta_1}+\nabla_{\bar\eta_1}]=0,\qquad [-\lambda_2\partial_{\eta_2}+\nabla_{\zeta_2},\ \lambda_2\partial_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,$
$[-\lambda_1\partial_{\eta_1}+\nabla_{\xi_1},\ -\lambda_2\partial_{\eta_2}+\nabla_{\zeta_2}]=[-\lambda_1\partial_{\eta_1}+\nabla_{\xi_1},\ \lambda_2\partial_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0,$
$[\lambda_1\partial_{\zeta_1}+\nabla_{\bar\eta_1},\ -\lambda_2\partial_{\eta_2}+\nabla_{\zeta_2}]=[\lambda_1\partial_{\zeta_1}+\nabla_{\bar\eta_1},\ \lambda_2\partial_{\bar\zeta_2}+\nabla_{\bar\eta_2}]=0.$

We will investigate the structure of the solution space of (3′). The system of eqs. (3′) is nothing but the integrability condition for the linear equations

(4) $(-\lambda_1\partial_{\eta_1}+\partial_{\bar\zeta_1}+A_{\bar\zeta_1})w(\lambda)=0,\qquad (\lambda_1\partial_{\zeta_1}+\partial_{\bar\eta_1}+A_{\bar\zeta_1})w(\lambda)=0,$
$(-\lambda_2\partial_{\eta_2}+\partial_{\zeta_2}+A_{\zeta_2})w(\lambda)=0,\qquad (\lambda_2\partial_{\bar\zeta_2}+\partial_{\bar\eta_2}+A_{\bar\eta_2})w(\lambda)=0.$

Proposition 1. *$A_{\bar\zeta_1}$, $A_{\bar\eta_1}$, A_{ζ_2}, $A_{\bar\eta_2}\in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1,\cdots,\bar\zeta_2]])$ are solutions of (3′) if and only if there exists a solution $w(\lambda)=\sum_{i,j\ge 0} w_{ij}\lambda_1^{-i}\lambda_2^{-j}$ of (4) such that $w_{0,0}=1$, namely, $w_{ij}\in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1,\cdots,\bar\zeta_2]])$ which satisfy $w_{0,0}=1$, $w_{ij}=0$ if $i<0$ or $j<0$, and*

(4′) $-\partial_{\eta_1}w_{i+1,j}+(\partial_{\bar\zeta_1}+A_{\bar\zeta_1})w_{ij}=0,\qquad \partial_{\zeta_1}w_{i+1,j}+(\partial_{\bar\eta_1}+A_{\bar\eta_1})w_{ij}=0,$
$-\partial_{\eta_2}w_{i,j+1}+(\partial_{\zeta_2}+A_{\zeta_2})w_{ij}=0,\qquad \partial_{\bar\zeta_2}w_{i,j+1}+(\partial_{\bar\eta_2}+A_{\bar\eta_2})w_{ij}=0$

for any $i, j\in \boldsymbol{Z}$.

When $i=j=0$, (4′) reads

(5) $-\partial_{\eta_1}w_{1,0}+A_{\bar\zeta_1}=0,\qquad \partial_{\zeta_1}w_{1,0}+A_{\bar\eta_1}=0,$
$-\partial_{\eta_2}w_{0,1}+A_{\zeta_2}=0,\qquad \partial_{\bar\zeta_2}w_{0,1}+A_{\bar\eta_2}=0.$

Therefore, to solve the eqs. (3′), it is sufficient to solve the equations

(4″) $-\partial_{\eta_1}w_{i+1,j}+\partial_{\bar\zeta_1}w_{ij}+(\partial_{\eta_1}w_{1,0})w_{ij}=0,$
$\partial_{\zeta_1}w_{i+1,j}+\partial_{\bar\eta_1}w_{ij}-(\partial_{\zeta_1}w_{1,0})w_{ij}=0,$
$-\partial_{\eta_2}w_{i,j+1}+\partial_{\zeta_2}w_{ij}+(\partial_{\eta_2}w_{0,1})w_{ij}=0,$
$\partial_{\bar\zeta_2}w_{i,j+1}+\partial_{\bar\eta_2}w_{ij}+(\partial_{\bar\zeta_2}w_{0,1})w_{ij}=0\qquad$ for any $i, j\in \boldsymbol{Z}$.

More precisely, we have

Proposition 2. *The relations (5) give a one-to-one correspondence between*

i) *solutions A of (3′)*

and

ii) *equivalence classes of the solutions $w(\lambda)$ of (4″) modulo right-multiplication by $v(\lambda)$ such that $(-\lambda_1\partial_{\eta_1}+\partial_{\bar\zeta_1})v(\lambda)=(\lambda_1\partial_{\zeta_1}+\partial_{\bar\eta_1})v(\lambda)=(-\lambda_2\partial_{\eta_2}+\partial_{\zeta_2})v(\lambda)=(\lambda_2\partial_{\bar\zeta_2}+\partial_{\bar\eta_2})v(\lambda)=0$, $v_{0,0}=1$, $v_{ij}=0$ if $i<0$, or $j<0$, $v_{ij}\in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1,\cdots,\bar\zeta_2]])$.*

§2. Motions on an infinite-dimensional Grassmann manifold. Let $I=\{(i,j)\in \mathrm{Z}\times\mathrm{Z}\mid i<0$ or $j<0\}$ and define a matrix of infinite size $\xi=(\xi^{ij}_{kl})^{(i,j)\in \mathrm{Z}\times\mathrm{Z}}_{(k,l)\in I}$ by the product of matrices $(w^*_{i-k,j-l})^{(i,j)\in\mathrm{Z}\times\mathrm{Z}}_{(k,l)\in I}$ and $(w_{i-k,j-l})^{(i,j)\in I}_{(k,l)\in I}$, i.e., by $\xi^{ij}_{kl}=\sum_{(g,h)\in I} w^*_{i-g,j-h}w_{g-k,h-l}$, where w^*_{ij} are coefficients of w^{-1}, i.e. $w^{-1}=\sum_{i,j\ge 0}w^*_{ij}\lambda_1^{-i}\lambda_2^{-j}$. Then we obtain $\xi^{ij}_{kl}=\delta^i_k\delta^j_l$ if $i<0$ or $j<0$, $\xi^{ij}_{kl}=0$ if $i<k$ or $j<l$, and $\Lambda_a\xi=\xi C_a$ $(a=1,2)$ where

$$\Lambda_1=(\delta^{i+1}_k\delta^j_l)^{(i,j)\in\mathrm{Z}\times\mathrm{Z}}_{(k,l)\in\mathrm{Z}\times\mathrm{Z}},\qquad \Lambda_2=(\delta^i_k\delta^{j+1}_l)^{(i,j)\in\mathrm{Z}\times\mathrm{Z}}_{(k,l)\in\mathrm{Z}\times\mathrm{Z}},$$
$$C_1=(\xi^{i+1,j}_{k,l})^{(i,j)\in I}_{(k,l)\in I},\qquad C_2=(\xi^{i,j+1}_{k,l})^{(i,j)\in I}_{(k,l)\in I}.$$

Here δ^i_k denotes the Kronecker's delta. Furthermore, the converse is true:

Proposition 3. *The above definition of ξ gives a one-to-one correspondence between*

i) $w(\lambda) \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \cdots, \bar{\zeta}_2]])[[\lambda_1^{-1}, \lambda_2^{-1}]]$ *such that* $w_{0,0} = 1$

and

ii) $\xi = (\xi_{kl}^{ij})_{(k,l) \in I}^{(i,j) \in \mathrm{Z} \times \mathrm{Z}}$ *such that* $\xi_{kl}^{ij} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \cdots, \bar{\zeta}_2]])$,

$$\begin{aligned} (6) \qquad & \xi_{k1}^{ij} = \delta_k^i \delta_l^j \text{ if } i<0 \text{ or } j<0, \qquad \xi_{kl}^{ij} = 0 \text{ if } i<k \text{ or } j<l, \\ & \Lambda_1 \xi = \xi C_1,\ \Lambda_2 \xi = \xi C_2 \qquad \text{for some } I \times I\text{-matrices } C_1, C_2, \end{aligned}$$

where the correspondence $\xi \to w$ *is defined by* $w_{ij} = -\xi_{-i,-j}^{0,0}$.

Therefore we can rewrite the equations $(4'')$:

Theorem 1. *Through the correspondence* $w \leftrightarrow \xi$, $(4'')$ *are equivalent to*

$$\begin{aligned} (7) \qquad & (-\Lambda_1 \partial_{\eta_1} + \partial_{\zeta_1}) \xi = \xi A_1, \qquad (\Lambda_1 \partial_{\bar{\zeta}_1} + \partial_{\bar{\eta}_1}) \xi = \xi B_1, \\ & (-\Lambda_2 \partial_{\eta_2} + \partial_{\zeta_2}) \xi = \xi A_2, \qquad (\Lambda_2 \partial_{\bar{\zeta}_2} + \partial_{\bar{\eta}_2}) \xi = \xi B_2, \end{aligned}$$

where A_1, A_2, B_1, B_2 *are* $I \times I$*-matrices uniquely determined by* ξ *if they exist.*

To investigate the structure of the solution space of (7), we consider an initial-value problem with respect to the subspace $\bar{\zeta}_1 = \bar{\eta}_1 = \zeta_2 = \bar{\eta}_2 = 0$. Unlike the case of self-dual Yang-Mills equations, we cannot solve it for arbitrary data; the data for which it is solvable must satisfy a system of differential equations. In fact, we have

Proposition 4. *The system of eqs.* (6) *and* (7) *implies*

$$\begin{aligned} (8) \qquad & \partial_{\alpha_0} \cdots \partial_{\alpha_i} \xi_{0l}^{i0} = 0 \qquad \text{for any } i \geq 0,\ \alpha_0, \cdots, \alpha_i = \eta_1, \zeta_1, \\ & \partial_{\beta_0} \cdots \partial_{\beta_j} \xi_{k0}^{0j} = 0 \qquad \text{for any } j \geq 0,\ \beta_0, \cdots, \beta_j = \eta_2, \bar{\zeta}_2. \end{aligned}$$

Conversely, for any initial datum satisfying (8) we can solve the initial-value problem:

Theorem 2. *For any*

$$\xi^{(0)} = (\xi^{(0)}{}_{kl}^{ij})_{(k,l) \in I}^{(i,j) \in \mathrm{Z} \times \mathrm{Z}}, \qquad \xi^{(0)}{}_{kl}^{ij} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \zeta_1, \eta_2, \bar{\zeta}_2]])$$

satisfying (6) *and* (8), *there exists a unique solution* ξ *to the initial-value problem, i.e.* $\xi = (\xi_{kl}^{ij})_{(k,l) \in I}^{(i,j) \in \mathrm{Z} \times \mathrm{Z}}$, $\xi_{kl}^{ij} \in \mathrm{gl}(n, \boldsymbol{C}[[\eta_1, \cdots, \bar{\zeta}_2]])$ *satisfying* (6), (7), *and* $\xi|_{\bar{\zeta}_1 = \bar{\eta}_1 = \zeta_2 = \bar{\eta}_2 = 0} = \xi^{(0)}$. *The solution* ξ *has the following form*:

$$\begin{aligned} & \xi = \tilde{\xi} (\tilde{\xi}^{(-)})^{-1}, \qquad \text{where} \\ & \tilde{\xi} = \exp(\bar{\zeta}_1 \Lambda_1 \partial_{\eta_1} - \bar{\eta}_1 \Lambda_1 \partial_{\zeta_2} + \zeta_2 \Lambda_2 \partial_{\eta_2} - \bar{\eta}_2 \Lambda_2 \partial_{\bar{\zeta}_2}) \xi^{(0)}, \\ & \tilde{\xi} = \begin{pmatrix} \tilde{\xi}^{(-)} \\ \tilde{\xi}^{(+)} \end{pmatrix}, \quad \tilde{\xi}^{(-)} = (\tilde{\xi}_{kl}^{ij})_{(k,l) \in I}^{(i,j) \in I}, \quad \tilde{\xi}^{(+)} = (\tilde{\xi}_{kl}^{ij})_{(k,l) \in I}^{(i,j) \in \mathrm{Z} \times \mathrm{Z} \setminus I}. \end{aligned}$$

In summary, by choosing the proper frame, the time evolutions in the initial-value problem can be regarded as evolutions generated by linear differential equations, and the solution space of $(4'')$ is faithfully parametrized by the solution space of equations (8) in the subspace $\bar{\zeta}_1 = \bar{\eta}_1 = \zeta_2 = \bar{\eta}_2 = 0$.

To solve the eqs. (8) is an open problem.

§ 3. Relation to the Yang-Mills fields.

Proposition 5. *If gauge fields* $\nabla, \tilde{\nabla}$ *satisfying* (1) *coincide as gauge fields in the diagonal subspace* Δ, *then* ∇ *and* $\tilde{\nabla}$ *are gauge-equivalent.*

Thanks to Proposition 5, we can see that the trivial extension of an (anti-) self-dual Yang-Mills field is the unique one up to gauge equivalence. Therefore, we have

Proposition 6. *The solutions of* (7) *which correspond to* (*anti-*) *self-dual Yang-Mills fields on Δ are characterized by*

$$\partial_{\eta_2}\partial_{\zeta_2}\xi^{0,0}_{0,-1}=0 \qquad (\partial_{\eta_1}\partial_{\zeta_1}\xi^{0,0}_{-1,0}=0).$$

All the (*anti-*) *self-dual Yang-Mills fields can be obtained in this way.*

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